Embedding theorems for Janelidze's matrix conditions

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Abstract

As a first objective, we characterise those essentially algebraic categories which satisfy properties like being unital, strongly unital, n-permutable, subtractive or protomodular. For each such property, we obtain a Mal'tsev condition as an equivalent condition. Using the language of Janelidze matrix conditions, we treat many of these properties together.

As a second objective, using these characterisations, we prove some embedding theorems for those properties in a regular context in the same style as we did in the companion paper [23]. Concrete examples of how to use these embedding theorems are given. Finally, to extend those embedding theorems to the exact context, we show that these properties are stable under the exact completion of a regular category.

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1 Introduction

By a Mal'tsev condition on a variety of universal algebras, it is commonly meant a condition which assume the existence of some terms in the corresponding theory satisfying some identities. The name originates from the most well-known example: a Mal'tsev variety is a variety for which the corresponding theory contains a ternary term $p(x, y, z)$ satisfying the identities $p(x, y, y) = x$ and $p(x, x, y) = y$. They have been characterised in [35] as those varieties for which the composition of congruences on each algebra is commutative, allowing a generalisation of the property for regular categories [14]. They have also been characterised in [32] as those varieties for which each homomorphic relation is difunctional, giving rise to a further generalisation in the finitely complete context [15].

Plenty of other examples can be found in the literature. For instance, the properties of being a unital [8], a strongly unital [8], an n-permutable [13], a subtractive [27] or a protomodular [7] category all give rise to a Mal'tsev condition in the algebraical context.

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In the series of paper [28, 29, 30, 31], Z. Janelidze proposed a common framework to study these properties. Many such properties can be represented by a matrix, whose columns display a categorical property on relations and whose rows carry out a Mal'tsev condition which characterises those algebraic varieties for which each homomorphic relation satisfies the property represented by the columns. One can get as examples the unital, the strongly unital, the n-permutable and the subtractive properties.

In the companion paper [23], we have characterised essentially algebraic categories which are Mal'tsev categories. As in the algebraic case, objects in essentially algebraic categories are given by an S-sorted set A (i.e., an object in Set^S, for a set S of sorts) endowed with operations $\prod_{i\in I} A_{s_i} \to A_s$ satisfying some given equations. The difference is that some of these operations can be only partially defined and defined exactly for those elements in the product satisfying some given equations involving totally defined operations. The characterisation of Mal'tsev essentially algebraic categories from [23] also gives rise to a Mal'tsev condition in the essentially algebraic world. The first aim of the present paper is to generalise this characterisation to the general form of conditions given by Janelidze matrices and to the protomodular condition. In each of these cases, we characterise essentially algebraic categories satisfying the given condition with a Mal'tsev condition. In the case of simple Janelidze matrices from [28], the corresponding Mal'tsev condition is very close to the algebraic one (see Theorem 3.4). However, in the general case, the corresponding Mal'tsev condition is a bit more complex (see Theorem 3.3). Note that our characterisations are given as Mal'tsev conditions, i.e., using terms and identities. This differs to what is generally done for locally presentable categories [18] (i.e., essentially algebraic categories). Indeed, in [6, 16, 20], the authors characterise those small finitely complete categories $\mathbb{C}$ for which the locally finitely presentable category $\mathrm{Lex}(\mathbb{C}, \mathrm{Set})$ has a given property. Here, $\mathrm{Lex}(\mathbb{C}, \mathrm{Set})$ denotes the category of finite limit preserving functors from $\mathbb{C}$ to Set, the category of sets.

The second main objective of this paper is to use these characterisations to build, for each of the above properties, a 'representative' regular category $\mathbb{M}$ having the considered property and such that each small regular category with the given property embeds 'nicely' in a power of $\mathbb{M}$. Here, 'nicely' means the embedding is a regular conservative faithful functor. We construct $\mathbb{M}$, in a similar way we did in [23] for the Mal'tsev case, i.e., as an essentially algebraic category. The proof of the embedding theorem is then also very similar, up to a few changes. We first need to prove that the properties we consider are stable under the free cofiltered limit completion. This completion is given, for a small finitely complete category $\mathbb{C}$, by the restricted Yoneda embedding $\mathbb{C} \hookrightarrow \mathrm{Lex}(\mathbb{C}, \mathrm{Set})^{\mathrm{op}}$. In that purpose, we prove that for a small regular category satisfying a given Janelidze matrix condition, its completion satisfies the same condition. For the protomodular case, we unfortunately need the existence of some colimits to make this step work, which will make the embedding theorem less powerful. The second step we need to achieve in order to prove our embedding theorems is to generalise the theory of approximate operations. In [10], the authors proved the existence of an approximate Mal'tsev co-operation on each object of a regular Mal'tsev category with binary coproducts. This has also been developed in the protomodular context in [11], for n-permutable categories in [36] and for simple Janelidze matrix conditions in [31]. We generalise it further here to the case of all Janelidze matrix conditions.

Due to these embedding theorems, if one wants to prove a categorical result (of a prescribed form) for all regular categories satisfying a Janelidze matrix condition, it is sufficient to prove it in the essentially algebraic representative category $\mathbb{M}$. We give concrete examples of how to use this technique to reduce the proof of categorical results to essentially algebraic arguments. As we are going to see, these proofs in the essentially algebraic world are mere translations of the corresponding proofs in the algebraic world. However, we provide an example where such a direct translation does not exist. This implies in particular that we have no hope to find a good embedding theorem for these properties for which the representing category $\mathbb{M}$ is an algebraic category.

We know from [19] that the exact completion [33] of a well-powered regular protomodular category is again protomodular. As a last part, we show that Janelidze matrix conditions are also stable under the exact completion of a well-powered regular category. We even show this result for properties involving implications of such matrix conditions. In view of this, our embedding theorems can be transposed in the exact context [3]. In particular, we can regularly and conservatively embed any small semi-abelian category [26] in a fixed exact homological category (in the sense of [5]).

The paper is divided as follows. In Section 2 we recall the background needed to understand the results. In particular, we recall the theory of Janelidze matrix conditions. In Section 3, we characterise those essentially algebraic categories satisfying a given Janelidze matrix condition, or being protomodular. Section 4 is devoted to the proof of our embedding theorems and Section 5 to their applications in concrete examples. We then treat the exact case in Section 6. Finally, we propose some open questions for further research in Section 7.

Remark 1.1. The reader is assumed to have read the companion paper [23] before this one. In particular, the theory of essentially algebraic categories is only briefly recalled here and we only show the main changes in the proof of the embedding theorems, since it is similar to the one in [23].

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2 Preliminaries

2.1 Essentially algebraic categories

Let us start by recalling the notion of an essentially algebraic category from [1]. Roughly speaking, this is a category of (many-sorted) algebraic structures with partial operations, for which the domain of definition of each of these partial operations is itself defined as the solution set of some system of totally defined equations. Such a category can be described by an *essentially algebraic theory*, that is a quintuple $\Gamma = (S, \Sigma, E, \Sigma_t, \mathrm{Def})$ where S is a set of sorts, Σ is an S-sorted signature of algebras, E is a set of Σ-equations, $\Sigma_t \subseteq \Sigma$ is the subset of 'total operation symbols' and Def is a function assigning to each operation symbol $\sigma\colon \prod_{i\in I} s_i \to s$ in $\Sigma \setminus \Sigma_t$ a set $\mathrm{Def}(\sigma)$ of Σ_t-equations in the variables x_i of sort s_i ($i \in I$). The notions of a Γ-model and of a homomorphism between Γ-models is then defined in the standard way (see [23] for more details). The category of Γ-models and their homomorphisms is denoted by $\mathrm{Mod}(\Gamma)$. They characterise, up to equivalence, all the locally presentable categories, also called the essentially algebraic categories [18, 1].

For Γ as above, if all arities of Σ are finite, if all equations of E use only a finite number of variables and if all sets $\mathrm{Def}(\sigma)$ are also finite, Γ is called a *finitary essentially algebraic* *theory*. A category which is equivalent to some category $\mathrm{Mod}(\Gamma)$ for a finitary essentially algebraic theory Γ is called a *finitary essentially algebraic category*, or, according to [18, 1], a locally finitely presentable category. The basic examples of finitary essentially algebraic categories are the finitary (many-sorted) quasivarieties and so, in particular, the finitary (many-sorted) varieties. The category Cat of small categories and Grpd of small groupoids are also finitary essentially algebraic.

For an essentially algebraic theory Γ, we recall that the category $\mathrm{Mod}(\Gamma)$ is complete and cocomplete and the forgetful functor $U\colon \mathrm{Mod}(\Gamma) \to \mathrm{Set}^S$ has a left adjoint. As described in [23], the free algebra on an S-sorted set of variables x_i of sort s_i (for any family $(s_i)_{i\in I}$ of sorts) is given, at a sort $s \in S$, by the set of equivalence classes of everywhere-defined terms $\tau\colon \prod_{i\in I} s_i \to s$ of Γ, where we identify the terms τ and τ' if and only if $\tau = \tau'$ is a theorem of Γ.

Let us also recall that $\mathrm{Mod}(\Gamma)$ possesses a (strong epi,mono)-factorisation system, since every homomorphism $f\colon A \to B$ factorises through its image $\mathrm{Im}(f)$ given by

$$\begin{aligned}\mathrm{Im}(f)_s = \{\tau((f(a_i))_{i\in I})\ | a_i \in A_{s_i} \text{ and } \tau\colon \prod_{i\in I} s_i \to s \text{ is a term of } \Sigma \text{ which}\\ \text{is defined in } B \text{ on } (f(a_i))_{i\in I}\},\end{aligned}$$

for each sort $s \in S$.

2.2 Some Mal'tsev conditions

In this subsection, we present some Mal'tsev conditions on algebraic varieties. These conditions assert the existence of some terms in the theory satisfying some identities. Often, they can be equivalently stated via a condition on homomorphic relations. This equivalent condition may then be stated in categorical terms which generalises the original algebraic definition. We recall here the 'Janelidze matrix conditions' introduced in [28, 29, 30, 31] which give a bunch of such examples, including the Mal'tsev [15], unital [8], strongly unital [8] and subtractive [27] conditions. These conditions are pictured by a matrix whose columns give the categorical condition on relations and whose rows represent the algebraic Mal'tsev condition. The condition of being a protomodular category [7] does not seem to be of that type, but however gives rise to a Mal'tsev condition in the algebraic world. We also recall this example in this subsection.

In order to recall the Janelidze matrix conditions, we first need to recall the notion of a $\mathcal{T}$-enrichment [17, 28]. Let $\mathcal{T}$ be a finitary one-sorted algebraic theory. An *internal $\mathcal{T}$-algebra* in a category $\mathbb{C}$ is an object A of $\mathbb{C}$ equipped with a structure of (ordinary) $\mathcal{T}$-algebra on $\mathrm{y}(A)$, where $\mathrm{y}\colon \mathbb{C} \to \mathrm{Set}^{\mathbb{C}^{\mathrm{op}}}$ is the Yoneda embedding. An *internal homomorphism of internal $\mathcal{T}$-algebras* is a morphism $f\colon A \to B$ in $\mathbb{C}$ such that $\mathrm{y}(f)$ is an ordinary homomorphism of $\mathcal{T}$-algebras. This forms the category $\mathrm{Alg}_{\mathcal{T}}\,\mathbb{C}$ of internal $\mathcal{T}$-algebras.

A *$\mathcal{T}$-enrichment* on $\mathbb{C}$ is a section of the forgetful functor $\mathrm{Alg}_{\mathcal{T}}\,\mathbb{C} \to \mathbb{C}$. In order words, it is the assignment of an internal $\mathcal{T}$-algebra structure on each object A of $\mathbb{C}$ in such a way that every morphism is an internal $\mathcal{T}$-algebra homomorphism. A *$\mathcal{T}$-enriched category* is a category $\mathbb{C}$ equipped with a fixed $\mathcal{T}$-enrichment. Thus, a $\mathcal{T}$-enriched category is a category $\mathbb{C}$ equipped with a factorisation $\mathrm{Hom}_{\mathbb{C}}$ of the usual functor $\hom_{\mathbb{C}}\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathrm{Set}$ through $U\colon \mathrm{Alg}_{\mathcal{T}} \to \mathrm{Set}$, the forgetful functor from the category of $\mathcal{T}$-algebras to the category of sets.

$$\begin{tikzcd} & \mathrm{Alg}_{\mathcal{T}} \arrow[d, "U"] \\ \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \arrow[ur, "\mathrm{Hom}_{\mathbb{C}}"] \arrow[r, "\mathrm{hom}_{\mathbb{C}}"'] & \mathrm{Set} \end{tikzcd}$$

A $\mathcal{T}$*-enriched functor* between the $\mathcal{T}$-enriched categories $\mathbb{C}$ and $\mathbb{D}$ is an ordinary functor $F\colon \mathbb{C} \to \mathbb{D}$ such that, for all $A, B \in \mathbb{C}$,

$$F\colon \mathrm{Hom}_{\mathbb{C}}(A,B) \to \mathrm{Hom}_{\mathbb{D}}(F(A),F(B))$$

is a homomorphism of $\mathcal{T}$-algebras.

If $\mathbb{C}$ has finite products, an internal $\mathcal{T}$-algebra structure on A is uniquely determined by, for each r-ary term τ of $\mathcal{T}$, the corresponding operation $\tau^A = \tau(p_1,\dots,p_r)\colon A^r \to A$ where $p_1,\dots,p_r$ are the product projections $A^r \to A$. Dually, if $\mathbb{C}$ has finite coproducts, a $\mathcal{T}$-enrichment on $\mathbb{C}$ is uniquely determined by the corresponding internal $\mathcal{T}$-co-algebra structure on each object A given by, for each r-ary term τ of $\mathcal{T}$, the co-operation $\tau^{A,\mathrm{op}} = \tau(\iota_1,\dots,\iota_r)\colon A \to rA$ where $\iota_1,\dots,\iota_r$ are the coproduct injections $A \to rA$.

Notice that, if $\mathbb{C}$ is a $\mathcal{T}$-enriched category, $\mathbb{C}^{\mathrm{op}}$ can be trivially provided with a $\mathcal{T}$-enrichment. Moreover, if $\mathbb{P}$ is a small category, then the equalities

$$\tau(\alpha_1,\dots,\alpha_r)_P = \tau(\alpha_{1,P},\dots,\alpha_{r,P})$$

for all r-ary terms τ of $\mathcal{T}$, object P of $\mathbb{P}$ and natural transformations $\alpha_1,\dots,\alpha_r\colon F \Rightarrow G$ define a $\mathcal{T}$-enrichment on the functor category $\mathbb{C}^{\mathbb{P}}$. If S is a set, we consider it as a discrete category to get a $\mathcal{T}$-enrichment on $\mathbb{C}^S$.

If $\mathcal{K}$ is another finitary one-sorted algebraic theory, $\mathcal{T}$-enrichments of $\mathrm{Alg}_{\mathcal{K}}$ are in one-to-one correspondence with central morphisms $\mathcal{T} \to \mathcal{K}$ of algebraic theories [17]. These are morphisms of algebraic theories such that for every term τ of $\mathcal{T}$, its interpretation τ^ι as a term of $\mathcal{K}$ commutes with every term υ of $\mathcal{K}$ in the sense that

$$\tau^\iota(\upsilon(x_{11},\dots,x_{1n}),\dots,\upsilon(x_{r1},\dots,x_{rn})) = \upsilon(\tau^\iota(x_{11},\dots,x_{r1}),\dots,\tau^\iota(x_{1n},\dots,x_{rn}))$$

is a theorem in $\mathcal{K}$ (where r and n are the arities of τ and υ respectively). The theory $\mathcal{T}$ is said to be *commutative* if the identity $\mathcal{T} \to \mathcal{T}$ is a central morphism, i.e., if every two operations in $\mathcal{T}$ commute with each other.

We can now recall the Janelidze matrix conditions from [28, 30]. Let $\mathcal{T}$ be a finitary one-sorted algebraic theory. An *extended matrix of terms in* $\mathcal{T}$ is a matrix

$$M = \left(\begin{array}{ccc|ccc} t_{11} & \cdots & t_{1b} & u_{11} & \cdots & u_{1b'} \\ \vdots & & \vdots & \vdots & & \vdots \\ t_{a1} & \cdots & t_{ab} & u_{a1} & \cdots & u_{ab'} \end{array}\right) \tag{1}$$

with $a \geqslant 1$, $b \geqslant 0$, $b' \geqslant 0$ and where the t_{ij}'s are terms of $\mathcal{T}$ in the variables $x_1,\dots,x_l$ and the u_{ij}'s are terms of $\mathcal{T}$ in the variables $x_1,\dots,x_k$ with $0 \leqslant l \leqslant k$. Let us denote by $X = \{x_{l+1},\dots,x_k\}$ the set of variables on which the u_{ij}'s depend but not the t_{ij}'s. To stress the fact that X might be not empty, we will denote this extended matrix by (M,X).

For an a-ary relation $r\colon R \rightarrowtail A^a$ in a regular $\mathcal{T}$-enriched category $\mathbb{C}$, we say that r is

(M,X)-*closed* if, when we consider the pullbacks

$$\begin{tikzcd}
P \arrow[r, "f'"] \arrow[d, tail, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & R^b \arrow[d, tail, "r^b"] \\
A^l \arrow[r, "\begin{pmatrix} t_{11}^A & \cdots & t_{1b}^A \\ \vdots & & \vdots \\ t_{a1}^A & \cdots & t_{ab}^A \end{pmatrix}"'] & (A^a)^b
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \arrow[r, "g'"] \arrow[d, tail, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & R^{b'} \arrow[d, tail, "r^{b'}"] \\
A^k \arrow[r, "\begin{pmatrix} u_{11}^A & \cdots & u_{1b'}^A \\ \vdots & & \vdots \\ u_{a1}^A & \cdots & u_{ab'}^A \end{pmatrix}"'] & (A^a)^{b'}
\end{tikzcd}$$

and

$$\begin{tikzcd}
T \arrow[r, tail, "t'"] \arrow[dd, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q \arrow[d, tail, "g"] \\
& A^k \cong A^l \times A^{k-l} \arrow[d, "\pi_1=(p_1,\ldots,p_l)"] \\
P \arrow[r, tail, "f"'] & A^l
\end{tikzcd} \tag{2}$$

then t is a regular epimorphism (or, in other words, f factors through the image of $\pi_1 g$). Here, $p_j\colon A^k \to A$ is the j^{th} projection for $1 \leqslant j \leqslant k$. We also have a description of (M,X)-closedness in terms of generalised elements as in the following proposition.

Proposition 2.1. Let (M,X) be an extended matrix of terms in the finitary one-sorted algebraic theory $\mathcal{T}$ as in (1). Let also $r\colon R \rightarrowtail A^a$ be an a-ary relation in the regular $\mathcal{T}$-enriched category $\mathbb{C}$. Then, r is (M,X)-closed if and only if for each morphism $y = (y_1,\ldots,y_l)\colon Y \to A^l$ in $\mathbb{C}$ such that

$$(t_{1j}(y_1,\ldots,y_l),\ldots,t_{aj}(y_1,\ldots,y_l))\colon Y \to A^a$$

factors through r for each $j \in \{1,\ldots,b\}$, there exists a regular epimorphism $p\colon Z \twoheadrightarrow Y$ and morphisms $z_{l+1},\ldots,z_k\colon Z \to A$ such that

$$(u_{1j}(y_1p,\ldots,y_lp,z_{l+1},\ldots,z_k),\ldots,u_{aj}(y_1p,\ldots,y_lp,z_{l+1},\ldots,z_k))\colon Z \to A^a$$

factors through r for each $j \in \{1,\ldots,b'\}$.

Proof. For the 'if part', it suffices to consider $y = f$. We then get a regular epimorphism $p\colon Z \twoheadrightarrow P$ and morphisms $z_{l+1},\ldots,z_k\colon Z \to A$. Considering the morphism

$$(p_1fp,\ldots,p_lfp,z_{l+1},\ldots,z_k)\colon Z \to A^k,$$

we get a morphism $z\colon Z \to Q$ such that

$$gz = (p_1fp,\ldots,p_lfp,z_{l+1},\ldots,z_k)$$

which implies $\pi_1 gz = fp$. Therefore, p factors through t and so t is a regular epimorphism.

Conversely, given such $y_1,\ldots,y_l\colon Y \to A$, let, for each $j \in \{1,\ldots,b\}$, $v_j\colon Y \to R$ be the unique morphism such that

$$rv_j = (t_{1j}(y_1,\ldots,y_l),\ldots,t_{aj}(y_1,\ldots,y_l)).$$

Let also $h\colon Y \to P$ be the unique morphism such that $fh = (y_1, \ldots, y_l)$ and $f'h = (v_1, \ldots, v_b)$. Eventually, we construct p as the pullback of t along h

$$\begin{tikzcd} Z \arrow[r, "p'"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "t"] \\ Y \arrow[r, "h"'] & P \end{tikzcd}$$

and $z_j = p_j g t' p'$ for $j \in \{l+1, \ldots, k\}$. It remains to see that

$$\begin{aligned} u_{ij}(y_1 p, \ldots, y_l p, z_{l+1}, \ldots, z_k) &= u_{ij}(p_1 fhp, \ldots, p_l fhp, p_{l+1} g t' p', \ldots, p_k g t' p') \\ &= u_{ij}(p_1 ftp', \ldots, p_l ftp', p_{l+1} g t' p', \ldots, p_k g t' p') \\ &= u_{ij}(p_1 g t' p', \ldots, p_k g t' p') \\ &= u_{ij}(p_1, \ldots, p_k) g t' p' \end{aligned}$$

for all $1 \leqslant i \leqslant a$ and $1 \leqslant j \leqslant b'$ and so

$$\begin{aligned} &(u_{1j}(y_1 p, \ldots, y_l p, z_{l+1}, \ldots, z_k), \ldots, u_{aj}(y_1 p, \ldots, y_l p, z_{l+1}, \ldots, z_k)) \\ &= (u_{1j}(p_1, \ldots, p_k), \ldots, u_{aj}(p_1, \ldots, p_k)) g t' p' \\ &= r p_j g' t' p' \end{aligned}$$

factors through r for each $j \in \{1, \ldots, b'\}$. $\square$

We say that the regular $\mathcal{T}$-enriched category $\mathbb{C}$ *has (M, X)-closed relations* if every a-ary relation $R \rightarrowtail A^a$ in $\mathbb{C}$ is (M, X)-closed. If $X = \varnothing$ and $b' = 1$, we recover the notion of a category with M-closed relations from [28].

The following theorem explains how the rows of M represent this condition as a Mal'tsev condition on algebraic varieties.

Theorem 2.2. (Corollary 3.2 in [30]) Consider a central morphism $\mathcal{T} \to \mathcal{K}$ of finitary one-sorted algebraic theories. Let (M, X) be an extended matrix of terms in $\mathcal{T}$ as in (1). Then, the regular $\mathcal{T}$-enriched category $\mathrm{Alg}_{\mathcal{K}}$ has (M, X)-closed relations if and only if there exist b-ary terms $p_1, \ldots, p_{b'}$ and l-ary terms $q_1, \ldots, q_{k-l}$ in $\mathcal{K}$ such that

$$p_j(t^\iota_{i1}(x_1, \ldots, x_l), \ldots, t^\iota_{ib}(x_1, \ldots, x_l)) = u^\iota_{ij}(x_1, \ldots, x_l, q_1(x_1, \ldots, x_l), \ldots, q_{k-l}(x_1, \ldots, x_l))$$

is a theorem of $\mathcal{K}$ for each $i \in \{1, \ldots, a\}$ and each $j \in \{1, \ldots, b'\}$, where t^ι denotes the interpretation in $\mathcal{K}$ of the term t in $\mathcal{T}$.

Example 2.3. If $\mathcal{T} = \mathrm{Th}[\mathrm{Set}]$ is the theory of sets and $\mathbb{C}$ is a regular category, then the following equivalences hold:

- $\mathbb{C}$ has $\left(\left(\begin{array}{ccc|c} x & y & y & x \\ x & x & y & y \end{array}\right), \varnothing\right)$-closed relations if and only if $\mathbb{C}$ is a Mal'tsev category [15, 28].
- More generally, for $n \geqslant 2$, $\mathbb{C}$ has (M, X)-closed relations for

$$(M, X) = \left(\left(\begin{array}{ccc|ccccc} x & y & y & x & z_1 & z_2 & \cdots & z_{n-2} \\ x & x & y & z_1 & z_2 & \cdots & z_{n-2} & y \end{array}\right), \{z_1, \ldots, z_{n-2}\}\right)$$

if and only if $\mathbb{C}$ is n-permutable. Indeed, having (M, X)-closed relations is equivalent to the condition that every binary relation R on A is such that

$$(1_A \wedge R)R^\circ(1_A \wedge R) \leqslant R^{n-1}$$

which is equivalent to being n-permutable [25] (see Section 5 for a recall on opposite and composition of relations in a regular category).

If $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$ is the theory of pointed sets and $\mathbb{C}$ a regular pointed category, then the following equivalences hold:

- $\mathbb{C}$ has $\left(\left(\begin{array}{cc|c} x & 0 & x \\ 0 & x & x \end{array}\right), \varnothing\right)$-closed relations if and only if $\mathbb{C}$ is unital [8, 28].
- $\mathbb{C}$ has $\left(\left(\begin{array}{ccc|c} x & 0 & 0 & x \\ y & y & x & x \end{array}\right), \varnothing\right)$-closed relations if and only if $\mathbb{C}$ is a strongly unital category [8, 28].
- $\mathbb{C}$ has $\left(\left(\begin{array}{cc|c} x & 0 & x \\ x & x & 0 \end{array}\right), \varnothing\right)$-closed relations if and only if $\mathbb{C}$ is subtractive [27, 28].

Although it seems that being a protomodular category cannot be expressed as a Janelidze matrix condition, this still induces a Mal'tsev condition on algebraic varieties, as attested by Theorem 2.4 below. We recall from [7] that a finitely complete category $\mathbb{C}$ is *protomodular* when the change of base functors of the fibration of points $\mathrm{Pt}(\mathbb{C}) \to \mathbb{C}$ reflect isomorphisms, or equivalently, when for each diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, shift right, "p"'] \arrow[dr, phantom, "\lrcorner"] & A' \arrow[d, shift right, "p'"'] \\ B \arrow[r, "v"'] \arrow[u, shift right, "s"'] & B' \arrow[u, shift right, "s'"'] \end{tikzcd}$$

with $p'u = vp$, $s'v = us$, $ps = 1_B$, $p's' = 1_{B'}$ and for which the square $p'u = vp$ is a pullback, the morphisms u and s' are jointly strongly epimorphic.

Theorem 2.4. (Theorem 1.1 in [9]) Let $\mathcal{K}$ be a finitary one-sorted algebraic theory. Then, the category $\mathrm{Alg}_{\mathcal{K}}$ is protomodular if and only if $\mathcal{K}$ contains, for some natural number $n \geqslant 0$,

- n constants $w_1, \ldots, w_n$,
- for each $1 \leqslant i \leqslant n$, a binary operation $d_i(x, y)$ such that $d_i(x, x) = w_i$ is a theorem of $\mathcal{K}$,
- an $n+1$-ary operation π such that $\pi(d_1(x, y), \ldots, d_n(x, y), y) = x$ is a theorem of $\mathcal{K}$.

3 Mal'tsev conditions in essentially algebraic categories

In this section, we characterise those essentially algebraic categories which satisfy the conditions described in Section 2.2. We separate it in two cases: first we treat the case of (M, X)-closed relations, and then we focus on the protomodular case. We will see that in both cases, the characterisation is expressed as the existence of terms in the theory satisfying some identities, and can therefore be considered as a Mal'tsev condition on essentially algebraic categories. For matrices with $k = l$ and for the protomodular case, these Mal'tsev conditions are quite closed to their algebraic version. It is however worthy to note that, for general Janelidze matrix conditions, they are more complex than their mere translation from the algebraic world to the essentially algebraic world.

We start by studying $\mathcal{T}$-enrichments of essentially algebraic categories. Let $\mathcal{T}$ be a finitary one-sorted algebraic theory and $\Gamma = (S, \Sigma, E, \Sigma_t, \mathrm{Def})$ an essentially algebraic theory. Suppose we have a $\mathcal{T}$-enrichment of $\mathrm{Mod}(\Gamma)$ and let τ be an r-ary operation symbol of $\mathcal{T}$. For a given $s \in S$, we consider the S-sorted set Y such that $Y_s = \{y_1, \ldots, y_r\}$ and $Y_{s'} = \varnothing$ for $s' \neq s$. The $\mathcal{T}$-algebra structure on $\mathrm{Fr}(Y)$ gives rise to an interpretation of τ into an everywhere-defined term $\tau^s \colon s^r \to s$ of Γ ($\tau^s = (\tau^{\mathrm{Fr}(Y)})_s(y_1, \ldots, y_r) \in \mathrm{Fr}(Y)_s$). Given now any Γ-model A and $a_1, \ldots, a_r \in A_s$, let $f \colon \mathrm{Fr}(Y) \to A$ be the unique Γ-homomorphism such that $f(y_i) = a_i$ for each $1 \leqslant i \leqslant r$. Then, since f is also an internal $\mathcal{T}$-homomorphism, the square

$$\begin{tikzcd}
\mathrm{Fr}(Y)^r \arrow[r, "\tau^{\mathrm{Fr}(Y)}"] \arrow[d, "f^r"'] & \mathrm{Fr}(Y) \arrow[d, "f"] \\
A^r \arrow[r, "\tau^A"'] & A
\end{tikzcd}$$

commutes. This implies that $(\tau^A)_s(a_1, \ldots, a_r) = \tau^s(a_1, \ldots, a_r)$. Hence, these interpretations turn axioms of $\mathcal{T}$ into theorems of Γ, in the sense that $\tau_1^s = \tau_2^s$ is a theorem of Γ for each axiom $\tau_1 = \tau_2$ of $\mathcal{T}$. Moreover, since $\tau^A \colon A^r \to A$ is a Γ-homomorphism for any Γ-model A,

$$\tau^s(\sigma((x_{1i})_{i\in I}), \ldots, \sigma((x_{ri})_{i\in I})) = \sigma((\tau^{s_i}(x_{1i}, \ldots, x_{ri}))_{i\in I}) \tag{3}$$

is a theorem of Γ for each operation symbol $\sigma \colon \prod_{i\in I} s_i \to s$ of Σ. These observations lead us to the following proposition. Compare this result with the description of $\mathcal{T}$-enrichments of algebraic categories as in Section 2.2.

Proposition 3.1. Let $\Gamma = (S, \Sigma, E, \Sigma_t, \mathrm{Def})$ be an essentially algebraic theory and $\mathcal{T}$ a finitary one-sorted algebraic theory. $\mathcal{T}$-enrichments of $\mathrm{Mod}(\Gamma)$ are in one-to-one correspondence with assignments, for all r-ary operation symbols τ of $\mathcal{T}$ and sort $s \in S$, of an everywhere-defined term $\tau^s \colon s^r \to s$ of Γ such that

- these interpretations turn axioms of $\mathcal{T}$ into theorems of Γ at each $s \in S$,
- for each operation symbol $\sigma \colon \prod_{i\in I} s_i \to s$ in Σ and operation symbol τ of $\mathcal{T}$, σ and τ commute, i.e., (3) is a theorem of Γ.

Proof. We proved above that each $\mathcal{T}$-enrichment gives rise to such an assignment. On the other hand, given such an assignment, we define an internal $\mathcal{T}$-algebra on every Γ-model A by letting $(\tau^A)_s = \tau^s \colon (A^r)_s = (A_s)^r \to A_s$ for all r-ary operation symbols τ of $\mathcal{T}$ and $s \in S$. It is routine verifications to check that this yields a Γ-homomorphism $\tau^A \colon A^r \to A$, that this gives rise to a $\mathcal{T}$-enrichment on $\mathrm{Mod}(\Gamma)$ and that these two applications are reciprocal inverses. $\square$

Before characterising essentially algebraic categories with (M, X)-closed relations, let us recall the characterisation of regular ones.

Proposition 3.2. [23, 22] Let Γ be an essentially algebraic theory. Then $\mathrm{Mod}(\Gamma)$ is a regular category if and only if, for each term $\theta \colon \prod_{i\in I} s_i \to s$ of Γ, there exists in Γ:

- a term $\pi \colon \prod_{j\in J} s'_j \to s$,

- for each $j \in J$, an everywhere-defined term $\alpha_j : s \to s'_j$ and
- for each $j \in J$, an everywhere-defined term $\mu_j : \prod_{i \in I} s_i \to s'_j$

such that

- $\pi((\alpha_j(x))_{j \in J})$ is an everywhere-defined term $s \to s$,
- $\pi((\alpha_j(x))_{j \in J}) = x$ is a theorem of Γ,
- $\alpha_j(\theta((x_i)_{i \in I})) = \mu_j((x_i)_{i \in I})$ is a theorem of Γ for each $j \in J$.

If Γ is a finitary essentially algebraic theory, it is enough to consider only finitary terms $\theta : \prod_{i=1}^{n} s_i \to s$.

We can now characterise essentially algebraic categories with (M, X)-closed relations.

Theorem 3.3. Let Γ be an essentially algebraic theory such that $\mathrm{Mod}(\Gamma)$ is regular, $\mathcal{T}$ a finitary one-sorted algebraic theory and (M, X) an extended matrix of terms in $\mathcal{T}$ as in (1). Given a $\mathcal{T}$-enrichment of $\mathrm{Mod}(\Gamma)$, this regular $\mathcal{T}$-enriched category $\mathrm{Mod}(\Gamma)$ has (M, X)-closed relations if and only if, for each sort $s \in S$, there exists in Γ:

- a term $\pi^s : \prod_{u \in U} s_u \to s$,
- for every $v \in \{1, \ldots, k\}$ and $u \in U$, an everywhere-defined term $q^u_v : s^l \to s_u$,
- for every $j \in \{1, \ldots, b'\}$ and $u \in U$, a term $p^u_j : s^b \to s_u$

such that

- the term
$$p^u_j(t^s_{i1}(x_1, \ldots, x_l), \ldots, t^s_{ib}(x_1, \ldots, x_l)) : s^l \to s_u$$
is everywhere-defined for all $i \in \{1, \ldots, a\}$, $j \in \{1, \ldots, b'\}$ and $u \in U$,
- the theorem
$$p^u_j(t^s_{i1}(x_1, \ldots, x_l), \ldots, t^s_{ib}(x_1, \ldots, x_l)) = u^{s_u}_{ij}(q^u_1(x_1, \ldots, x_l), \ldots, q^u_k(x_1, \ldots, x_l))$$
holds in Γ for all $i \in \{1, \ldots, a\}$, $j \in \{1, \ldots, b'\}$ and $u \in U$,
- the term
$$\pi^s((q^u_v(x_1, \ldots, x_l))_{u \in U}) : s^l \to s$$
is everywhere-defined for each $v \in \{1, \ldots, l\}$,
- the theorem
$$\pi^s((q^u_v(x_1, \ldots, x_l))_{u \in U}) = x_v$$
holds in Γ for each $v \in \{1, \ldots, l\}$.

Proof. Suppose firstly that such terms are given. Let $R \subseteq A^a$ be an a-ary relation on A in $\mathrm{Mod}(\Gamma)$. Let P and Q be as in the definition of (M, X)-closedness. We have to prove that $f : P \rightarrowtail A^l$ factors through the image of $\pi_1 g : Q \to A^l$. So, let $s \in S$ and $(a_1, \ldots, a_l) \in P_s \subseteq A^l_s$. Thus, we know that

$$(t^s_{1j}(a_1, \ldots, a_l), \ldots, t^s_{aj}(a_1, \ldots, a_l)) \in R_s$$

for all $j \in \{1, \ldots, b\}$. So, for all $j \in \{1, \ldots, b'\}$ and $u \in U$,

$$\begin{aligned}
&\Big(u_{1j}^{s_u}(q_1^u(a_1, \ldots, a_l), \ldots, q_k^u(a_1, \ldots, a_l)), \ldots, u_{aj}^{s_u}(q_1^u(a_1, \ldots, a_l), \ldots, q_k^u(a_1, \ldots, a_l))\Big) \\
&= \big(p_j^u(t_{11}^s(a_1, \ldots, a_l), \ldots, t_{1b}^s(a_1, \ldots, a_l)), \ldots, p_j^u(t_{a1}^s(a_1, \ldots, a_l), \ldots, t_{ab}^s(a_1, \ldots, a_l))\big) \\
&\in R_{s_u}.
\end{aligned}$$

This means that

$$b_u = (q_1^u(a_1, \ldots, a_l), \ldots, q_k^u(a_1, \ldots, a_l)) \in Q_{s_u}$$

for all $u \in U$. Therefore,

$$\begin{aligned}
(a_1, \ldots, a_l) &= (\pi^s((q_1^u(a_1, \ldots, a_l))_{u \in U}), \ldots, \pi^s((q_l^u(a_1, \ldots, a_l))_{u \in U})) \\
&= \pi^s(((q_1^u(a_1, \ldots, a_l), \ldots, q_l^u(a_1, \ldots, a_l)))_{u \in U}) \\
&= \pi^s((\pi_1 g(b_u))_{u \in U}) \\
&\in \mathrm{Im}(\pi_1 g)_s
\end{aligned}$$

and R is (M, X)-closed.

Conversely, let us suppose that $\mathrm{Mod}(\Gamma)$ has (M, X)-closed relations. Let $s \in S$ and Y be the S-sorted set such that $Y_s = \{y_1, \ldots, y_l\}$ and $Y_{s'} = \varnothing$ for $s' \neq s$. We denote by R the smallest a-ary homomorphic relation on $\mathrm{Fr}(Y)$ such that

$$(t_{1j}^s(y_1, \ldots, y_l), \ldots, t_{aj}^s(y_1, \ldots, y_l)) \in R_s$$

for all $j \in \{1, \ldots, b\}$. It is easy to prove that for each $s' \in S$,

$$\begin{aligned}
R_{s'} = \{&a\text{-tuple of everywhere-defined terms} \\
&(\tau(t_{11}^s(y_1, \ldots, y_l), \ldots, t_{1b}^s(y_1, \ldots, y_l)), \ldots, \tau(t_{a1}^s(y_1, \ldots, y_l), \ldots, t_{ab}^s(y_1, \ldots, y_l))) \quad | \\
&\tau\colon s^b \to s' \text{ is a term of } \Gamma\} \\
\subseteq{}& \mathrm{Fr}(Y)_{s'}^a.
\end{aligned}$$

Let P and Q be as in the definition of (M, X)-closedness for R. Since $(y_1, \ldots, y_l) \in P_s$ and R is (M, X)-closed, $(y_1, \ldots, y_l) \in \mathrm{Im}(\pi_1 g)_s$. Thus, there exists a term $\pi^s\colon \prod_{u \in U} s_u \to s$ and an element $q^u \in Q_{s_u}$ for each $u \in U$ such that

$$\pi^s((\pi_1 g(q^u))_{u \in U}) = (y_1, \ldots, y_l).$$

By construction of Q, for each $u \in U$, we know there exist everywhere-defined terms $q_1^u, \ldots, q_k^u\colon s^l \to s_u$ such that

$$(u_{1j}^{s_u}(q_1^u, \ldots, q_k^u), \ldots, u_{aj}^{s_u}(q_1^u, \ldots, q_k^u)) \in R_{s_u}$$

for all $j \in \{1, \ldots, b'\}$ and the term

$$\pi^s((q_v^u(y_1, \ldots, y_l))_{u \in U})$$

is everywhere-defined and equal to y_v for all $v \in \{1, \ldots, l\}$. The above description of R_{s_u} gives the terms $p_j^u\colon s^b \to s_u$ for every $u \in U$ and $j \in \{1, \ldots, b'\}$ with the required properties. $\square$

In the case where $k = l$, we actually do not need to consider the term π^s.

Theorem 3.4. Let Γ be an essentially algebraic theory such that $\mathrm{Mod}(\Gamma)$ is regular, $\mathcal{T}$ a finitary one-sorted algebraic theory and (M, X) an extended matrix of terms in $\mathcal{T}$ as in (1) such that $k = l$. Given a $\mathcal{T}$-enrichment of $\mathrm{Mod}(\Gamma)$, this regular $\mathcal{T}$-enriched category $\mathrm{Mod}(\Gamma)$ has (M, X)-closed relations if and only if, for each sort $s \in S$, there exist terms $p_1, \ldots, p_{b'} \colon s^b \to s$ in Γ such that

- the term
$$p_j(t^s_{i1}(x_1, \ldots, x_l), \ldots, t^s_{ib}(x_1, \ldots, x_l)) \colon s^l \to s$$
is everywhere-defined for each $i \in \{1, \ldots, a\}$ and each $j \in \{1, \ldots, b'\}$,
- the theorem
$$p_j(t^s_{i1}(x_1, \ldots, x_l), \ldots, t^s_{ib}(x_1, \ldots, x_l)) = u^s_{ij}(x_1, \ldots, x_l)$$
holds in Γ for each $i \in \{1, \ldots, a\}$ and each $j \in \{1, \ldots, b'\}$.

Proof. Let us suppose first that $\mathrm{Mod}(\Gamma)$ has (M, X)-closed relations and let us consider the terms given by Theorem 3.3. For $s \in S$ and $j \in \{1, \ldots, b'\}$, we define the term $p_j \colon s^b \to s$ as
$$p_j(y_1, \ldots, y_b) = \pi^s((p^u_j(y_1, \ldots, y_b))_{u \in U}).$$
Let $i \in \{1, \ldots, a\}$. We already know that the term $p^u_j(t^s_{i1}(x_1, \ldots, x_l), \ldots, t^s_{ib}(x_1, \ldots, x_l))$ is everywhere-defined for each $u \in U$. Moreover, to prove that
$$\pi^s((p^u_j(t^s_{i1}(x_1, \ldots, x_l), \ldots, t^s_{ib}(x_1, \ldots, x_l)))_{u \in U})$$
is everywhere-defined, it is equivalent to prove that
$$\pi^s((u^{s_u}_{ij}(q^u_1(x_1, \ldots, x_l), \ldots, q^u_l(x_1, \ldots, x_l)))_{u \in U})$$
is everywhere-defined. But this is the case since $\pi^s((q^u_v(x_1, \ldots, x_l))_{u \in U})$ is everywhere-defined for each $v \in \{1, \ldots, l\}$ (using the facts that $k = l$ and u_{ij} commutes with terms of Γ). Moreover, we have the following theorems in Γ:
$$\begin{aligned}
&p_j(t^s_{i1}(x_1, \ldots, x_l), \ldots, t^s_{ib}(x_1, \ldots, x_l))\\
&= \pi^s((p^u_j(t^s_{i1}(x_1, \ldots, x_l), \ldots, t^s_{ib}(x_1, \ldots, x_l)))_{u \in U})\\
&= \pi^s((u^{s_u}_{ij}(q^u_1(x_1, \ldots, x_l), \ldots, q^u_l(x_1, \ldots, x_l)))_{u \in U})\\
&= u^s_{ij}(\pi^s((q^u_1(x_1, \ldots, x_l))_{u \in U}), \ldots, \pi^s((q^u_l(x_1, \ldots, x_l))_{u \in U}))\\
&= u^s_{ij}(x_1, \ldots, x_l)
\end{aligned}$$
proving the required properties of p_j.

Conversely, suppose that, for each sort s, such terms $p_1, \ldots, p_{b'} \colon s^b \to s$ exist. Then, to get the terms required by Theorem 3.3, it suffices to set $\pi^s = 1_s$ and $q^u_v(x_1, \ldots, x_l) = x_v$ for each $v \in \{1, \ldots, l\}$. □

Remark 3.5. Extended matrices (M, X) for which $k = l$ were already considered in the first paper [28] of the corresponding series of papers. In this case, the corresponding categorical property makes sense for an arbitrary finitely complete category (not necessarily regular). Using a proof similar to the one used for Theorem 3.3, one can remove the regularity assumption in Theorem 3.4. This has been done in [22].

Let us conclude this section with the characterisation of protomodular essentially algebraic categories. Compare it with Theorem 2.4.

Theorem 3.6. Let Γ be an essentially algebraic theory. Then $\mathrm{Mod}(\Gamma)$ is protomodular if and only if, for each sort $s \in S$, there exists in Γ

- a term $\pi^s \colon \left(\prod_{i \in I} s_i\right) \times s \to s$,
- for each $i \in I$, an everywhere-defined term $d_i \colon s^2 \to s_i$,
- for each $i \in I$, an everywhere-defined constant term w_i of sort s_i

such that

- $d_i(x, x) = w_i$ is a theorem of Γ for each $i \in I$,
- the term
$$\pi^s((d_i(x, y))_{i \in I}, y) \colon s^2 \to s$$
is everywhere-defined,
- the theorem
$$\pi^s((d_i(x, y))_{i \in I}, y) = x$$
holds in Γ.

Proof. Firstly, let us suppose the conditions in the statement hold in Γ and let us prove that $\mathrm{Mod}(\Gamma)$ is protomodular. So, we consider a morphism of points in the fibre over B in $\mathrm{Mod}(\Gamma)$, i.e., a diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, shift left, "p"'] & & A' \arrow[dl, shift left, "q"] \\ & B \arrow[ul, shift left, "t"] \arrow[ur, shift left, "u"'] & \end{tikzcd}$$

with $pt = 1_B = qu$, $qf = p$ and $ft = u$. We also consider a morphism $v \colon B' \to B$ such that the image f' of f by the change of base functor along v

$$\begin{tikzcd} A \times_{p,v} B' \arrow[rr, "f'"] \arrow[dr, shift left, "p'"'] & & A' \times_{q,v} B' \arrow[dl, shift left, "q'"] \\ & B' \arrow[ul, shift left, "t'"] \arrow[ur, shift left, "u'"'] & \end{tikzcd}$$

is an isomorphism. We have to prove that f is also an isomorphism. Let us first prove it is a monomorphism. So, let $s \in S$ and $a, a' \in A_s$ be such that $f(a) = f(a')$. We consider also the terms given in the statement for s. For each $i \in I$, we have

$$p(d_i(a, a')) = q(d_i(f(a), f(a'))) = q(d_i(f(a), f(a))) = q(w_i) = w_i$$

and $(d_i(a, a'), w_i) \in (A \times_{p,v} B')_{s_i}$. Moreover,

$$f'(d_i(a, a'), w_i) = (f(d_i(a, a')), w_i) = (d_i(f(a), f(a')), w_i) = (w_i, w_i) = f'(w_i, w_i)$$

and $d_i(a, a') = w_i = d_i(a', a')$ since f'_{s_i} is injective. Therefore, we have

$$a = \pi^s((d_i(a, a'))_{i \in I}, a') = \pi^s((d_i(a', a'))_{i \in I}, a') = a'$$

and f_s is injective. Now, we show that $\mathrm{Im}(f)_s = A'_s$. So, let $c \in A'_s$. For each $i \in I$, we know that

$$q(d_i(c, ftq(c))) = d_i(q(c), qftq(c)) = d_i(q(c), q(c)) = w_i$$

from which $(d_i(c, ftq(c)), w_i) \in (A' \times_{q,v} B')_{s_i}$. Since f'_{s_i} is bijective, there exists an element $a_i \in A_{s_i}$ such that $(a_i, w_i) \in (A \times_{p,v} B')_{s_i}$ (i.e., $p(a_i) = w_i$) and $f(a_i) = d_i(c, ftq(c))$. Therefore, we can say that

$$c = \pi^s((d_i(c, ftq(c)))_{i \in I}, ftq(c)) = \pi^s((f(a_i))_{i \in I}, ftq(c)) \in \mathrm{Im}(f)_s$$

and f is an isomorphism.

Conversely, let us suppose that $\mathrm{Mod}(\Gamma)$ is protomodular and let $s \in S$. Let also X and Y be the S-sorted sets defined by $X_s = \{x_1, x_2\}$, $Y_s = \{y\}$ and $X_{s'} = \varnothing = Y_{s'}$ for all $s' \neq s$. We consider the diagram

$$\begin{tikzcd}
 & \mathrm{Fr}(X) \times_{p,!} \mathrm{Fr}(\varnothing) \arrow[r, "p_2"] \arrow[dl, two heads] \arrow[dd, "p_1"'] \arrow[ddr, phantom, "\lrcorner", very near start] & \mathrm{Fr}(\varnothing) \arrow[dd, "!"] \\
\mathrm{Im}(p_1) \arrow[dr, hook] & & \\
 & \mathrm{Fr}(X) \arrow[r, two heads, shift right, "p"'] & \mathrm{Fr}(Y) \arrow[l, shift right, "t"']
\end{tikzcd}$$

where the square is a pullback and p and t are defined by $p(x_1) = p(x_2) = y$ and $t(y) = x_2$. Since $pt = 1_{\mathrm{Fr}(Y)}$, t is a monomorphism and we can see $\mathrm{Fr}(Y)$ as a submodel of $\mathrm{Fr}(X)$. We write $\mathrm{Im}(p_1) \vee \mathrm{Fr}(Y)$ for the smallest submodel of $\mathrm{Fr}(X)$ which contains $\mathrm{Im}(p_1) \cup \mathrm{Fr}(Y)$. It is routine to prove it is described by

$$\begin{aligned}
&(\mathrm{Im}(p_1) \vee \mathrm{Fr}(Y))_{s'} = \\
&\Bigg\{ \tau((p_1(z_i))_{i \in I}, x_2) \,|\, \tau \colon \left(\prod_{i \in I} s_i \right) \times s \to s' \text{ is a term of } \Gamma, z_i \in (\mathrm{Fr}(X) \times_{p,!} \mathrm{Fr}(\varnothing))_{s_i} \\
&\text{and } \tau((p_1(z_i))_{i \in I}, x_2) \text{ is defined in } \mathrm{Fr}(X) \Bigg\}
\end{aligned}$$

for all $s' \in S$. We have thus a morphism of points in the fibre over $\mathrm{Fr}(Y)$ in $\mathrm{Mod}(\Gamma)$:

$$\begin{tikzcd}
\mathrm{Im}(p_1) \vee \mathrm{Fr}(Y) \arrow[rr, hook, "i"] \arrow[dr, shift right, "pi"'] & & \mathrm{Fr}(X) \arrow[dl, two heads, shift left, "p"] \\
 & \mathrm{Fr}(Y) \arrow[ul, shift right, "t"'] \arrow[ur, shift left, "t"] &
\end{tikzcd}$$

By construction, its image by the change of base functor along the unique morphism $! \colon \mathrm{Fr}(\varnothing) \to \mathrm{Fr}(Y)$ is the pullback of i along p_1, which is an isomorphism since p_1 factors through i. Moreover, since $\mathrm{Mod}(\Gamma)$ is protomodular, we know that i is an isomorphism as well and $x_1 \in (\mathrm{Im}(p_1) \vee \mathrm{Fr}(Y))_s$. In view of the description of $(\mathrm{Im}(p_1) \vee \mathrm{Fr}(Y))_s$, we have a term

$$\pi^s \colon \left(\prod_{i \in I} s_i \right) \times s \to s$$

and elements $z_i \in (\mathrm{Fr}(X) \times_{p,!} \mathrm{Fr}(\varnothing))_{s_i}$ (for $i \in I$) such that

$$\pi^s((p_1(z_i))_{i \in I}, x_2)$$

is defined in $\mathrm{Fr}(X)$ and equal to x_1. Now, considering the description of the pullback $\mathrm{Fr}(X) \times_{p,!} \mathrm{Fr}(\varnothing)$, there exists, for each $i \in I$, an everywhere-defined term $d_i \colon s^2 \to s_i$ and an everywhere-defined constant term w_i of sort s_i such that $z_i = (d_i, w_i)$ and $p(d_i) = !(w_i)$. We thus got all the terms and theorems we were looking for. $\square$

Remark 3.7. In the algebraic case, the Mal'tsev condition of being protomodular seems to be of another type than the ones corresponding to Janelidze matrix conditions. Indeed, in Theorem 2.4, the arity of the term π is not fixed and the number of terms d_i and w_i depends on this arity, while in Theorem 2.2 both the number of terms and the arities are fixed. In the essentially algebraic case, this difference disappear for general Janelidze matrix conditions and protomodularity, as we can see from Theorems 3.3 and 3.6.

4 Embedding theorems

As we did in [23] for the Mal'tsev case, we are now going to prove some embedding theorems for the categorical conditions considered in Section 2.2. The idea is to construct a fixed category $\mathbb{M}$ having the considered property, in such a way that each small category with that property has a 'good' embedding in a power of $\mathbb{M}$. With such an embedding theorem, we can reduce the proof of many categorical statements for all categories with the given property to the proof of its validity in $\mathbb{M}$. We will construct $\mathbb{M}$ as an essentially algebraic category using Theorems 3.3 and 3.6. The corresponding embedding theorems will be proved using the same technique as in [23], i.e., using approximate co-operations and the free cofiltered limit completion of a category. We will only present explicitly the parts of the proof which cannot be adapted on the spot from the proof in [23]. Let us start with the construction of $\mathbb{M}$, for the property of having (M, X)-closed relations. We will suppose $\mathcal{T}$ to be commutative.

4.1 The representing category $\mathrm{Mod}(\Gamma_{(M,X)})$

Firstly, if Γ and Γ' are two essentially algebraic theories, we will write $\Gamma \subseteq \Gamma'$ to mean that $S \subseteq S'$, $\Sigma \subseteq \Sigma'$, $E \subseteq E'$, $\Sigma_t \subseteq \Sigma'_t$, $\Sigma \setminus \Sigma_t \subseteq \Sigma' \setminus \Sigma'_t$ and $\mathrm{Def}(\sigma) = \mathrm{Def}'(\sigma)$ for all $\sigma \in \Sigma \setminus \Sigma_t$. In this case, we have a forgetful functor $U\colon \mathrm{Mod}(\Gamma') \to \mathrm{Mod}(\Gamma)$.

Now, let us fix a commutative finitary one-sorted algebraic theory $\mathcal{T}$ and an extended matrix (M, X) of terms in $\mathcal{T}$ as in (1). We write $\Sigma_r^{\mathcal{T}}$ for the set of r-ary operation symbols of $\mathcal{T}$. We are going to construct recursively a series of finitary essentially algebraic theories

$$\Gamma^0 \subseteq \Delta^1 \subseteq \cdots \subseteq \Gamma^n \subseteq \Delta^{n+1} \subseteq \cdots$$

and a $\mathcal{T}$-enrichment on the $\mathrm{Mod}(\Gamma^n)$'s. Let us first define $\Gamma^0 = (S^0, \Sigma^0, E^0, \Sigma_t^0, \mathrm{Def}^0)$:

- $S^0 = \{\star\}$,
- $\Sigma^0 = \Sigma_t^0 = \{\tau^\star\colon \star^r \to \star \,|\, r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}\}$,
- $E^0 = \{$all axioms from $\mathcal{T}$ for the $\tau^\star$'s$\}$.

We consider the obvious $\mathcal{T}$-enrichment on $\mathrm{Mod}(\Gamma^0) \cong \mathrm{Alg}_{\mathcal{T}}$. Now, let us suppose we have defined

$$\Gamma^0 \subseteq \Delta^1 \subseteq \cdots \subseteq \Delta^n \subseteq \Gamma^n$$

and the $\mathcal{T}$-enrichment on $\mathrm{Mod}(\Gamma^n)$ (with $\Gamma^n = (S^n, \Sigma^n, E^n, \Sigma_t^n, \mathrm{Def}^n)$). We are going to construct

$$\Delta^{n+1} = (S'^{n+1}, \Sigma'^{n+1}, E'^{n+1}, \Sigma_t'^{n+1}, \mathrm{Def}'^{n+1})$$

first. Below, $\overline{S}^0 = S^0$ and $\overline{S}^n = S^n \setminus S^{n-1}$ if $n \geqslant 1$.

$$S'^{n+1} = S^n \cup \{(s,0), (s,1) \,|\, s \in \overline{S}^n\} \cong S^n \sqcup \overline{S}^n \sqcup \overline{S}^n,$$

$$\begin{aligned}
\Sigma_t^{\prime n+1} = \Sigma_t^n &\cup \{\tau^{(s,0)}\colon (s,0)^r \to (s,0) \,|\, r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\} \\
&\cup \{\tau^{(s,1)}\colon (s,1)^r \to (s,1) \,|\, r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\} \\
&\cup \{\alpha^s\colon s \to (s,0) \,|\, s \in \overline{S}^n\} \\
&\cup \{\rho_1^s, \ldots, \rho_{b'}^s\colon s^b \to (s,0) \,|\, s \in \overline{S}^n\} \\
&\cup \{\kappa_1^s, \ldots, \kappa_{k-l}^s\colon s^l \to (s,0) \,|\, s \in \overline{S}^n\} \\
&\cup \{\eta^s, \varepsilon^s\colon (s,0) \to (s,1) \,|\, s \in \overline{S}^n\},
\end{aligned}$$

$$\Sigma^{\prime n+1} = \Sigma^n \cup \Sigma_t^{\prime n+1} \cup \{\pi^s\colon (s,0) \to s \,|\, s \in \overline{S}^n\},$$

$$\begin{aligned}
E^{\prime n+1} = E^n \cup \{&\rho_j^s(t_{i1}^s(x_1, \ldots, x_l), \ldots, t_{ib}^s(x_1, \ldots, x_l)) \\
&= u_{ij}^{(s,0)}(\alpha^s(x_1), \ldots, \alpha^s(x_l), \kappa_1^s(x_1, \ldots, x_l), \ldots, \kappa_{k-l}^s(x_1, \ldots, x_l)) \,| \\
&\qquad\qquad\qquad\qquad 1 \leqslant i \leqslant a, 1 \leqslant j \leqslant b', s \in \overline{S}^n\} \\
\cup \{&\eta^s(\alpha^s(x)) = \varepsilon^s(\alpha^s(x)) \,|\, s \in \overline{S}^n\} \\
\cup \{&\pi^s(\alpha^s(x)) = x \,|\, s \in \overline{S}^n\} \\
\cup \{&\alpha^s(\pi^s(x)) = x \,|\, s \in \overline{S}^n\} \\
\cup \{&\text{all axioms from } \mathcal{T} \text{ for the } \tau^{(s,0)}\text{'s and the } \tau^{(s,1)}\text{'s} \,|\, s \in \overline{S}^n\} \\
\cup \{&\tau^{(s,0)}(\alpha^s(x_1), \ldots, \alpha^s(x_r)) = \alpha^s(\tau^s(x_1, \ldots, x_r)) \,|\, r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\} \\
\cup \{&\tau^{(s,0)}(\rho_j^s(x_{11}, \ldots, x_{1b}), \ldots, \rho_j^s(x_{r1}, \ldots, x_{rb})) \\
&= \rho_j^s(\tau^s(x_{11}, \ldots, x_{r1}), \ldots, \tau^s(x_{1b}, \ldots, x_{rb})) \,| \\
&\qquad\qquad\qquad\qquad 1 \leqslant j \leqslant b', r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\} \\
\cup \{&\tau^{(s,0)}(\kappa_v^s(x_{11}, \ldots, x_{1l}), \ldots, \kappa_v^s(x_{r1}, \ldots, x_{rl})) \\
&= \kappa_v^s(\tau^s(x_{11}, \ldots, x_{r1}), \ldots, \tau^s(x_{1l}, \ldots, x_{rl})) \,| \\
&\qquad\qquad\qquad\qquad 1 \leqslant v \leqslant k-l, r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\} \\
\cup \{&\tau^{(s,1)}(\eta^s(x_1), \ldots, \eta^s(x_r)) = \eta^s(\tau^{(s,0)}(x_1, \ldots, x_r)) \,|\, r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\} \\
\cup \{&\tau^{(s,1)}(\varepsilon^s(x_1), \ldots, \varepsilon^s(x_r)) = \varepsilon^s(\tau^{(s,0)}(x_1, \ldots, x_r)) \,|\, r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\} \\
\cup \{&\tau^s(\pi^s(x_1), \ldots, \pi^s(x_r)) = \pi^s(\tau^{(s,0)}(x_1, \ldots, x_r)) \,|\, r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\}
\end{aligned}$$

and

$$\begin{cases} \mathrm{Def}^{\prime n+1}(\sigma) = \mathrm{Def}^n(\sigma) \ \text{ if } \ \sigma \in \Sigma^n \setminus \Sigma_t^n \\ \mathrm{Def}^{\prime n+1}(\pi^s) = \{\eta^s(x) = \varepsilon^s(x)\} \ \text{ for } \ s \in \overline{S}^n. \end{cases}$$

Hence, we have $\Gamma^n \subseteq \Delta^{n+1}$ and we consider the obvious $\mathcal{T}$-enrichment on $\mathrm{Mod}(\Delta^{n+1})$.

Let now T^{n+1} be the set of finitary terms $\theta\colon \prod_{i=1}^m s_i \to s$ of $\Sigma^{\prime n+1}$ which are not terms of $\Sigma^{\prime n}$ (where we consider $\Sigma^{\prime 0} = \varnothing$). We then define Γ^{n+1} as:

$$S^{n+1} = S^{\prime n+1} \cup \{s_\theta, s'_\theta \,|\, \theta \in T^{n+1}\} \cong S^{\prime n+1} \sqcup T^{n+1} \sqcup T^{n+1},$$

$$
\begin{aligned}
\Sigma_t^{n+1} = \Sigma_t'^{n+1} &\cup \{\tau^{s_\theta}\colon s_\theta^r \to s_\theta \mid r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, \theta \in T^{n+1}\} \\
&\cup \{\tau^{s'_\theta}\colon (s'_\theta)^r \to s'_\theta \mid r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, \theta \in T^{n+1}\} \\
&\cup \{\alpha_\theta\colon s \to s_\theta \mid \theta\colon \prod_{i=1}^m s_i \to s \in T^{n+1}\} \\
&\cup \{\mu_\theta\colon \prod_{i=1}^m s_i \to s_\theta \mid \theta\colon \prod_{i=1}^m s_i \to s \in T^{n+1}\} \\
&\cup \{\eta_\theta, \varepsilon_\theta\colon s_\theta \to s'_\theta \mid \theta \in T^{n+1}\},
\end{aligned}
$$

$$
\Sigma^{n+1} = \Sigma'^{n+1} \cup \Sigma_t^{n+1} \cup \{\pi_\theta\colon s_\theta \to s \mid \theta\colon \prod_{i=1}^m s_i \to s \in T^{n+1}\},
$$

$$
\begin{aligned}
E^{n+1} = E'^{n+1} &\cup \{\eta_\theta(\alpha_\theta(x)) = \varepsilon_\theta(\alpha_\theta(x)) \mid \theta \in T^{n+1}\} \\
&\cup \{\pi_\theta(\alpha_\theta(x)) = x \mid \theta \in T^{n+1}\} \\
&\cup \{\alpha_\theta(\pi_\theta(x)) = x \mid \theta \in T^{n+1}\} \\
&\cup \{\alpha_\theta(\theta(x_1, \dots, x_m)) = \mu_\theta(x_1, \dots, x_m) \mid \theta\colon \prod_{i=1}^m s_i \to s \in T^{n+1}\} \\
&\cup \{\text{all axioms from } \mathcal{T} \text{ for the } \tau^{s_\theta}\text{'s and the } \tau^{s'_\theta}\text{'s} \mid \theta \in T^{n+1}\} \\
&\cup \{\tau^{s_\theta}(\alpha_\theta(x_1), \dots, \alpha_\theta(x_r)) = \alpha_\theta(\tau^s(x_1, \dots, x_r)) \mid \\
&\qquad\qquad\qquad r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, \theta\colon \prod_{i=1}^m s_i \to s \in T^{n+1}\} \\
&\cup \{\tau^{s_\theta}(\mu_\theta(x_{11}, \dots, x_{1m}), \dots, \mu_\theta(x_{r1}, \dots, x_{rm})) \\
&\quad = \mu_\theta(\tau^{s_1}(x_{11}, \dots, x_{r1}), \dots, \tau^{s_m}(x_{1m}, \dots, x_{rm})) \mid \\
&\qquad\qquad\qquad r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, \theta\colon \prod_{i=1}^m s_i \to s \in T^{n+1}\} \\
&\cup \{\tau^{s'_\theta}(\eta_\theta(x_1), \dots, \eta_\theta(x_r)) = \eta_\theta(\tau^{s_\theta}(x_1, \dots, x_r)) \mid r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, \theta \in T^{n+1}\} \\
&\cup \{\tau^{s'_\theta}(\varepsilon_\theta(x_1), \dots, \varepsilon_\theta(x_r)) = \varepsilon_\theta(\tau^{s_\theta}(x_1, \dots, x_r)) \mid r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, \theta \in T^{n+1}\} \\
&\cup \{\tau^{s}(\pi_\theta(x_1), \dots, \pi_\theta(x_r)) = \pi_\theta(\tau^{s_\theta}(x_1, \dots, x_r)) \mid \\
&\qquad\qquad\qquad r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, \theta\colon \prod_{i=1}^m s_i \to s \in T^{n+1}\}
\end{aligned}
$$

and

$$
\begin{cases}
\operatorname{Def}^{n+1}(\sigma) = \operatorname{Def}'^{n+1}(\sigma) & \text{if } \sigma \in \Sigma'^{n+1} \setminus \Sigma_t'^{n+1} \\
\operatorname{Def}^{n+1}(\pi_\theta) = \{\eta_\theta(x) = \varepsilon_\theta(x)\} & \text{for } \theta \in T^{n+1}.
\end{cases}
$$

Thus, we have $\Delta^{n+1} \subseteq \Gamma^{n+1}$ and we consider the obvious $\mathcal{T}$-enrichment on $\operatorname{Mod}(\Gamma^{n+1})$. We have constructed

$$
\Gamma^0 \subseteq \Delta^1 \subseteq \Gamma^1 \subseteq \cdots
$$

Let $\Gamma_{(M,X)}$ be the union of these finitary essentially algebraic theories. By that we obviously mean $S_{(M,X)} = \bigcup_{n \geqslant 0} S^n$, $\Sigma_{(M,X)} = \bigcup_{n \geqslant 0} \Sigma^n$, $E_{(M,X)} = \bigcup_{n \geqslant 0} E^n$, $\Sigma_{t,(M,X)} = \bigcup_{n \geqslant 0} \Sigma_t^n$ and $\operatorname{Def}_{(M,X)}(\sigma) = \operatorname{Def}^n(\sigma)$ for all $n \geqslant 0$ and $\sigma \in \Sigma^n \setminus \Sigma_t^n$. We provide $\operatorname{Mod}(\Gamma_{(M,X)})$ with the $\mathcal{T}$-enrichment coming from the $\mathcal{T}$-enrichments on the $\operatorname{Mod}(\Gamma^n)$'s.

Proposition 4.1. Let $\mathcal{T}$ be a commutative finitary one-sorted algebraic theory and (M,X) an extended matrix of terms in $\mathcal{T}$. Then the $\mathcal{T}$-enriched category $\mathrm{Mod}(\Gamma_{(M,X)})$ is regular with (M,X)-closed relations.

Proof. It is the 'Γ ingredient' of the construction which makes $\mathrm{Mod}(\Gamma_{(M,X)})$ a regular category. Indeed, each finitary term θ of $\Sigma_{(M,X)}$ is in T^{n+1} for some $n \geqslant 0$, which makes the conditions of Proposition 3.2 hold.

On the other hand, the 'Δ part' of the construction ensures that $\mathrm{Mod}(\Gamma_{(M,X)})$ has (M,X)-closed relations. To see that, it suffices to use Theorem 3.3 with the terms $\pi^s\colon (s,0) \to s$, $\alpha^s \circ p_1, \dots, \alpha^s \circ p_l, \kappa^s_1, \dots, \kappa^s_{k-l}\colon s^l \to (s,0)$ (where $p_1, \dots, p_l\colon s^l \to s$ are the projections), and $\rho^s_1, \dots, \rho^s_{b'}\colon s^b \to (s,0)$. □

4.2 The embedding $\mathbb{C} \hookrightarrow \mathrm{Lex}(\mathbb{C}, \mathrm{Set})^{\mathrm{op}}$

As in [4, 23], a key ingredient to prove the embedding theorem is to consider (the restriction of) the Yoneda embedding $i\colon \mathbb{C} \hookrightarrow \mathrm{Lex}(\mathbb{C}, \mathrm{Set})^{\mathrm{op}}$ for a small finitely complete category $\mathbb{C}$. Here, $\mathrm{Lex}(\mathbb{C}, \mathrm{Set})$ is the category of finite limit preserving functors $\mathbb{C} \to \mathrm{Set}$ and its dual $\mathrm{Lex}(\mathbb{C}, \mathrm{Set})^{\mathrm{op}}$ is denoted by $\widetilde{\mathbb{C}}$ as in [4]. Due to the embedding $i\colon \mathbb{C} \hookrightarrow \widetilde{\mathbb{C}}$, we treat $\mathbb{C}$ as a full subcategory of $\widetilde{\mathbb{C}}$.

Theorem 4.2. [2, 18] Let $\mathbb{C}$ be a small finitely complete category. The following statements hold.

1. $\widetilde{\mathbb{C}}$ is complete and cocomplete.

2. In $\widetilde{\mathbb{C}}$, cofiltered limits commute with limits and finite colimits.

3. The embedding $i\colon \mathbb{C} \hookrightarrow \widetilde{\mathbb{C}}$ preserves all colimits and finite limits.

4. For all $A \in \widetilde{\mathbb{C}}$, the span $(A, (c)_{(C,c) \in (A \downarrow i)})$ is the limit of the functor
$$\begin{aligned} (A \downarrow i) &\longrightarrow \widetilde{\mathbb{C}} \\ c\colon A \to i(C) &\longmapsto i(C). \end{aligned}$$

5. The embedding $i\colon \mathbb{C} \hookrightarrow \widetilde{\mathbb{C}}$ is the free cofiltered limit completion of $\mathbb{C}$.

Now, if we are given a $\mathcal{T}$-enrichment on $\mathbb{C}$ for a finitary one-sorted algebraic theory $\mathcal{T}$, we construct a $\mathcal{T}$-enrichment on $\widetilde{\mathbb{C}}$ in the following way. If A is an object of $\widetilde{\mathbb{C}}$ and τ an r-ary term of $\mathcal{T}$, we define $\tau^A\colon A^r \to A$ as the unique morphism which makes the diagram

$$\begin{tikzcd} A^r \arrow[rr, "\tau^A"] \arrow[d, "c^r"'] & & A \arrow[d, "c"] \\ i(C)^r \arrow[r, "\cong"'] & i(C^r) \arrow[r, "i(\tau^C)"'] & i(C) \end{tikzcd}$$

commute for all $(C,c) \in (A \downarrow i)$. This morphism exists and is unique by Theorem 4.2.4. This makes $\widetilde{\mathbb{C}}$ a $\mathcal{T}$-enriched category and i a $\mathcal{T}$-enriched functor.

In [4] and [23], the embedding $\mathbb{C} \hookrightarrow \widetilde{\mathbb{C}}$ was used because, in a regular context, each object of $\widetilde{\mathbb{C}}$ has a regular $\mathbb{C}$-projective cover. This has been proved in [21] in the abelian context and in [4] in the regular context.

Theorem 4.3. [4] Let $\mathbb{C}$ be a small regular category. Then $\widetilde{\mathbb{C}}$ is regular and each object $X \in \widetilde{\mathbb{C}}$ admits a regular $\mathbb{C}$-projective cover, i.e., a regular epimorphism $\widehat{X} \twoheadrightarrow X$ where $\widehat{X}$ is a regular $\mathbb{C}$-projective object.

We thus have an embedding of $\mathbb{C}$ in a larger category $\widetilde{\mathbb{C}}$ with a regular $\mathbb{C}$-projective covering. To be able to use it, we need to prove that this $\widetilde{\mathbb{C}}$ shares the same properties as $\mathbb{C}$. In particular, in [23], a crucial step was to prove that if $\mathbb{C}$ is a small regular Mal'tsev category, then $\widetilde{\mathbb{C}}$ is also a regular Mal'tsev category. We now prove the corresponding result for the property of being a regular $\mathcal{T}$-enriched category with (M, X)-closed relations. This has been (or will be respectively) proved in [22] and [24] using the theory of unconditional exactness properties. Since this is a central ingredient in the proof of our embedding theorem, we sketch a direct proof here.

Proposition 4.4. [24, 22] Let $\mathcal{T}$ be a commutative finitary one-sorted algebraic theory, (M, X) an extended matrix of terms in $\mathcal{T}$ and $\mathbb{C}$ a small regular $\mathcal{T}$-enriched category with (M, X)-closed relations. Then, $\widetilde{\mathbb{C}}$ is also a regular $\mathcal{T}$-enriched category with (M, X)-closed relations.

Proof. We already know from [4] that $\widetilde{\mathbb{C}}$ is a regular category. To prove it has (M, X)-closed relations, it is equivalent to show that for any family of a parallel maps

$$\begin{tikzcd} R \arrow[r, shift left=2, "r_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "r_a"'] & A \end{tikzcd}$$

in $\widetilde{\mathbb{C}}$, the morphism $t\colon T \to P$ constructed as in the definition of (M, X)-closed relations from $r = (r_1, \dots, r_a)\colon R \to A^a$ (see diagram (2)) is a regular epimorphism. From [34], we know that this family of parallel maps $(r_1, \dots, r_a)$ in $\widetilde{\mathbb{C}}$ is a cofiltered limit of diagrams of the same shape from $\mathbb{C}$.

$$\begin{tikzcd} & & R \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] \arrow[dll, dotted] \arrow[dr, dotted] & A \arrow[dll, dotted] \arrow[dr, dotted] & \\ i(R_n) \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & i(A_n) & \cdots & i(R_{n'}) \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & i(A_{n'}) \end{tikzcd}$$

Since (cofiltered) limits commute with limits, the morphism $t\colon T \to P$ constructed from the original diagram in $\widetilde{\mathbb{C}}$ as in diagram (2) is the cofiltered limit of the morphisms $i(t_n)\colon i(T_n) \to i(P_n)$ constructed from the diagrams in $\mathbb{C}$. But $\mathbb{C}$ has (M, X)-closed relations, so these t_n's are all regular epimorphisms. Since cofiltered limits in $\widetilde{\mathbb{C}}$ commute with finite colimits (Theorem 4.2.2), this implies that t is also a regular epimorphism and $\widetilde{\mathbb{C}}$ has (M, X)-closed relations. $\square$

4.3 Approximate co-solutions and the embedding theorem

The last ingredient used in the proof of the embedding theorem in [23] was to use the approximate Mal'tsev co-operations in $\widetilde{\mathbb{C}}$ introduced in [10]. This concept has been generalised in [36] for n-permutable categories and in [31] for simple Janelidze matrix conditions, i.e., the one corresponding to an extended matrix (M, X) with $k = l$ and $b' = 1$. In order to prove our embedding theorem in full generality, we generalise here this concept to the case of any extended matrix.

Let (M, X) be an extended matrix of terms in the finitary one-sorted algebraic theory $\mathcal{T}$ as in (1) and $\mathbb{C}$ a regular $\mathcal{T}$-enriched category with finite coproducts. An *approximate* *co-solution for* (M, X) *on* $Y \in \mathbb{C}$ is a span

$$\begin{tikzcd} & Z \arrow[dl, "(p_1,\dots,p_{b'})"'] \arrow[d, "d"] \arrow[dr, "(q_1,\dots,q_{k-l})"] & \\ (bY)^{b'} & Y & (lY)^{k-l} \end{tikzcd}$$

satisfying

$$\begin{pmatrix} t_{i1}(\iota_1,\dots,\iota_l) \\ \vdots \\ t_{ib}(\iota_1,\dots,\iota_l) \end{pmatrix} p_j = u_{ij}(\iota_1 d, \dots, \iota_l d, q_1, \dots, q_{k-l}) \colon Z \to lY$$

for all $i \in \{1, \dots, a\}$ and $j \in \{1, \dots, b'\}$, where $\iota_1, \dots, \iota_l \colon Y \to lY$ are the coproduct injections. The morphism $d \colon Z \to Y$ is then called the *approximation* of the approximate co-solution. For each $Y \in \mathbb{C}$, there is a universal approximate co-solution on Y. Let L be the following product,

$$\begin{tikzcd} & L = (bY)^{b'} \times Y \times (lY)^{k-l} \arrow[dl, "(p'_1,\dots,p'_{b'})"'] \arrow[d, "d'"] \arrow[dr, "(q'_1,\dots,q'_{k-l})"] & \\ (bY)^{b'} & Y & (lY)^{k-l} \end{tikzcd}$$

consider the equaliser

$$\begin{tikzcd}[column sep=huge] W(Y) \arrow[r, tail, "e"] & L \arrow[r, shift left, "\left(\begin{pmatrix} t_{i1}(\iota_1,\dots,\iota_l) \\ \vdots \\ t_{ib}(\iota_1,\dots,\iota_l) \end{pmatrix} p'_j\right)_{\substack{i\in\{1,\dots,a\} \\ j\in\{1,\dots,b'\}}}"] \arrow[r, shift right, "\left(u_{ij}(\iota_1 d',\dots,\iota_l d', q'_1,\dots,q'_{k-l})\right)_{\substack{i\in\{1,\dots,a\} \\ j\in\{1,\dots,b'\}}}"'] & (lY)^{a\times b'} \end{tikzcd}$$

and let $d^Y = d'e$, $(p^Y_1, \dots, p^Y_{b'}) = (p'_1 e, \dots, p'_{b'} e)$ and $(q^Y_1, \dots, q^Y_{k-l}) = (q'_1 e, \dots, q'_{k-l} e)$. Then

$$\begin{tikzcd} & W(Y) \arrow[dl, "(p^Y_1,\dots,p^Y_{b'})"'] \arrow[d, "d^Y"] \arrow[dr, "(q^Y_1,\dots,q^Y_{k-l})"] & \\ (bY)^{b'} & Y & (lY)^{k-l} \end{tikzcd}$$

is the *universal approximate co-solution for* (M, X) *on* Y, in the sense that any other factorises uniquely through it.

Theorem 4.5. Let $\mathcal{T}$ be a finitary one-sorted algebraic theory, (M, X) an extended matrix of terms in $\mathcal{T}$ and $\mathbb{C}$ a regular $\mathcal{T}$-enriched category with finite coproducts. The following statements are equivalent:

1. $\mathbb{C}$ has (M, X)-closed relations.

2. For each $Y \in \mathbb{C}$, there is an approximate co-solution for (M, X) on Y for which the approximation d is a regular epimorphism.

3. For each $Y \in \mathbb{C}$, the universal approximate co-solution for (M, X) on Y is such that the approximation d^Y is a regular epimorphism.

Proof. 3 $\Rightarrow$ 2 is obvious and 2 $\Rightarrow$ 3 follows from the universality of $W(Y)$. Let us prove 1 $\Rightarrow$ 2. So let $Y \in \mathbb{C}$ and consider the pullbacks

$$\begin{tikzcd}
P \arrow[r, "f'"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & (bY)^b \arrow[d, "{\begin{pmatrix} t_{11}^{Y,\mathrm{op}} & \cdots & t_{a1}^{Y,\mathrm{op}} \\ \vdots & & \vdots \\ t_{1b}^{Y,\mathrm{op}} & \cdots & t_{ab}^{Y,\mathrm{op}} \end{pmatrix}^b}"] \\
(lY)^l \arrow[r, "{\begin{pmatrix} t_{11}^{lY} & \cdots & t_{1b}^{lY} \\ \vdots & & \vdots \\ t_{a1}^{lY} & \cdots & t_{ab}^{lY} \end{pmatrix}}"'] & ((lY)^a)^b
\end{tikzcd}$$

$$\begin{tikzcd}
Q \arrow[r, "g'"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & (bY)^{b'} \arrow[d, "{\begin{pmatrix} t_{11}^{Y,\mathrm{op}} & \cdots & t_{a1}^{Y,\mathrm{op}} \\ \vdots & & \vdots \\ t_{1b}^{Y,\mathrm{op}} & \cdots & t_{ab}^{Y,\mathrm{op}} \end{pmatrix}^{b'}}"] \\
(lY)^k \arrow[r, "{\begin{pmatrix} u_{11}^{lY} & \cdots & u_{1b'}^{lY} \\ \vdots & & \vdots \\ u_{a1}^{lY} & \cdots & u_{ab'}^{lY} \end{pmatrix}}"'] & ((lY)^a)^{b'}
\end{tikzcd}$$

and

$$\begin{tikzcd}
T \arrow[r, "t'"] \arrow[dd, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q \arrow[d, "g"] \\
 & (lY)^k \cong (lY)^l \times (lY)^{k-l} \arrow[d, "\pi_1"] \\
P \arrow[r, "f"'] & (lY)^l.
\end{tikzcd}$$

Since the image of the morphism

$$\begin{tikzcd}
bY \arrow[r, "{\begin{pmatrix} t_{11}^{Y,\mathrm{op}} & \cdots & t_{a1}^{Y,\mathrm{op}} \\ \vdots & & \vdots \\ t_{1b}^{Y,\mathrm{op}} & \cdots & t_{ab}^{Y,\mathrm{op}} \end{pmatrix}}"] & (lY)^a
\end{tikzcd}$$

is an (M, X)-closed relation, t is a regular epimorphism. In addition, the diagram

$$\begin{tikzcd}
Y \arrow[r, "{(\iota_1,\ldots,\iota_b)}"] \arrow[d, "{(\iota_1,\ldots,\iota_l)}"'] & (bY)^b \arrow[d, "{\begin{pmatrix} t_{11}^{Y,\mathrm{op}} & \cdots & t_{a1}^{Y,\mathrm{op}} \\ \vdots & & \vdots \\ t_{1b}^{Y,\mathrm{op}} & \cdots & t_{ab}^{Y,\mathrm{op}} \end{pmatrix}^b}"] \\
(lY)^l \arrow[r, "{\begin{pmatrix} t_{11}^{lY} & \cdots & t_{1b}^{lY} \\ \vdots & & \vdots \\ t_{a1}^{lY} & \cdots & t_{ab}^{lY} \end{pmatrix}}"'] & ((lY)^a)^b
\end{tikzcd}$$

commutes and $(\iota_1,\ldots,\iota_l)$ factors through f. Hence, if we consider the pullback

$$\begin{tikzcd} S \arrow[r, "s'"] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q \arrow[d, "g"] \\ & (lY)^k \cong (lY)^l \times (lY)^{k-l} \arrow[d, "\pi_1"] \\ Y \arrow[r, "(\iota_1,\ldots,\iota_l)"'] & (lY)^l \end{tikzcd}$$

s is a pullback of t and is thus a regular epimorphism. Therefore, we have the expected approximate co-solution for (M,X) on Y:

$$\begin{tikzcd} & S \arrow[dl, "g's'"'] \arrow[d, "s"] \arrow[dr, "\pi_2 gs'"] & \\ (bY)^{b'} & Y & (lY)^{k-l} \end{tikzcd}$$

It is actually an approximate co-solution for (M,X) on Y since, for each $i \in \{1,\ldots,a\}$ and $j \in \{1,\ldots,b'\}$,

$$\begin{aligned} \begin{pmatrix} t_{i1}(\iota_1,\ldots,\iota_l) \\ \vdots \\ t_{ib}(\iota_1,\ldots,\iota_l) \end{pmatrix} g'_j s' &= u_{ij}^{lY} gs' \\ &= u_{ij}(g_1 s',\ldots,g_k s') \\ &= u_{ij}(\iota_1 s,\ldots,\iota_l s, g_{l+1}s',\ldots,g_k s') \end{aligned}$$

by definition of the pullback Q, where $g=(g_1,\ldots,g_k)$ and $g'=(g'_1,\ldots,g'_{b'})$.

It remains to prove $3 \Rightarrow 1$. Let $r\colon R \rightarrowtail A^a$ be an a-ary relation on A in $\mathbb{C}$. We are going to use Proposition 2.1 to prove that r is (M,X)-closed. So, we consider a morphism $y=(y_1,\ldots,y_l)\colon Y \to A^l$ such that

$$(t_{1j}(y_1,\ldots,y_l),\ldots,t_{aj}(y_1,\ldots,y_l))\colon Y \to A^a$$

factors through r for each $j \in \{1,\ldots,b\}$. By assumption, we have a regular epimorphism $d^Y\colon W(Y) \twoheadrightarrow Y$ and morphisms

$$z_v = \begin{pmatrix} y_1 \\ \vdots \\ y_l \end{pmatrix} q^Y_{v-l}\colon W(Y) \to A$$

for $v \in \{l+1,\ldots,k\}$. Now, for each $j \in \{1,\ldots,b'\}$,

$$\begin{aligned} &(u_{1j}(y_1 d^Y,\ldots,y_l d^Y, z_{l+1},\ldots,z_k),\ldots,u_{aj}(y_1 d^Y,\ldots,y_l d^Y,z_{l+1},\ldots,z_k)) \\ &= \left(\begin{pmatrix} y_1 \\ \vdots \\ y_l \end{pmatrix} u_{1j}(\iota_1 d^Y,\ldots,\iota_l d^Y, q^Y_1,\ldots,q^Y_{k-l}),\ldots, \begin{pmatrix} y_1 \\ \vdots \\ y_l \end{pmatrix} u_{aj}(\iota_1 d^Y,\ldots,\iota_l d^Y,q^Y_1,\ldots,q^Y_{k-l}) \right) \\ &= \left(\begin{pmatrix} y_1 \\ \vdots \\ y_l \end{pmatrix} \begin{pmatrix} t^{Y,\mathrm{op}}_{11} \\ \vdots \\ t^{Y,\mathrm{op}}_{1b} \end{pmatrix} p_j,\ldots, \begin{pmatrix} y_1 \\ \vdots \\ y_l \end{pmatrix} \begin{pmatrix} t^{Y,\mathrm{op}}_{a1} \\ \vdots \\ t^{Y,\mathrm{op}}_{ab} \end{pmatrix} p_j \right) \\ &= \begin{pmatrix} t_{11}(y_1,\ldots,y_l) & \cdots & t_{a1}(y_1,\ldots,y_l) \\ \vdots & & \vdots \\ t_{1b}(y_1,\ldots,y_l) & \cdots & t_{ab}(y_1,\ldots,y_l) \end{pmatrix} p_j \end{aligned}$$

which factors through r by assumption on $y_1,\ldots,y_l$. This proves that r is (M,X)-closed. □

We can now state our embedding theorem. What remains to be done to prove it is just a mere translation of the proof in [23] for the Mal'tsev case. Therefore, we omit it (a detailed proof can still be found in [22]).

Theorem 4.6. Let $\mathcal{T}$ be a commutative finitary one-sorted algebraic theory, (M, X) an extended matrix of terms in $\mathcal{T}$ and $\mathbb{C}$ a small regular $\mathcal{T}$-enriched category with (M, X)-closed relations. We denote by $\mathrm{Sub}(1)$ the set of subobjects of the terminal object 1 of $\mathbb{C}$. Then, there exists a faithful $\mathcal{T}$-enriched embedding $\phi\colon \mathbb{C} \hookrightarrow \mathrm{Mod}(\Gamma_{(M,X)})^{\mathrm{Sub}(1)}$ which preserves and reflects finite limits, isomorphisms and regular epimorphisms. Moreover, for each morphism $f\colon C \to C'$ in $\mathbb{C}$, each $I \in \mathrm{Sub}(1)$ and each $s \in S_{(M,X)}$,

$$(\mathrm{Im}\,\phi(f)_I)_s = \{(\phi(f)_I)_s(x) \mid x \in (\phi(C)_I)_s\}.$$

4.4 Embedding theorem for regular protomodular categories

We now turn our attention to the protomodular case. In a similar way than we did for Janelidze matrix conditions, we are going to prove an embedding theorem for regular protomodular categories and for *homological categories* [5] (i.e., pointed regular protomodular categories). Unfortunately, this time we will need to assume the existence of some colimits. In order to treat both cases (pointed and non-pointed) simultaneously, we are going to consider regular $\mathcal{T}$-enriched protomodular categories. The non-pointed case then corresponds to $\mathcal{T} = \mathrm{Th}[\mathrm{Set}]$ while the pointed case corresponds to $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$. So let $\mathcal{T}$ be a commutative finitary one-sorted algebraic theory and let us construct the essentially algebraic theory $\Gamma^{\mathcal{T}}_{\mathrm{proto}}$. Again, $\Sigma^{\mathcal{T}}_r$ represents the set of r-ary operation symbols of $\mathcal{T}$.

We are again going to construct recursively a series of finitary essentially algebraic theories

$$\Gamma^0 \subseteq \Delta^1 \subseteq \cdots \subseteq \Gamma^n \subseteq \Delta^{n+1} \subseteq \cdots$$

and a $\mathcal{T}$-enrichment on the corresponding categories of models. Let us first define $\Gamma^0 = (S^0, \Sigma^0, E^0, \Sigma^0_t, \mathrm{Def}^0)$:

- $S^0 = \{\star\}$,
- $\Sigma^0 = \Sigma^0_t = \{\tau^\star\colon \star^r \to \star \mid r \geqslant 0, \tau \in \Sigma^{\mathcal{T}}_r\}$,
- $E^0 = \{$all axioms from $\mathcal{T}$ for the $\tau^\star$'s$\}$.

We consider the obvious $\mathcal{T}$-enrichment on $\mathrm{Mod}(\Gamma^0) \cong \mathrm{Alg}_{\mathcal{T}}$. Now, let us suppose we have defined

$$\Gamma^0 \subseteq \Delta^1 \subseteq \cdots \subseteq \Delta^n \subseteq \Gamma^n$$

and the $\mathcal{T}$-enrichment on $\mathrm{Mod}(\Gamma^n)$ (with $\Gamma^n = (S^n, \Sigma^n, E^n, \Sigma^n_t, \mathrm{Def}^n)$). We are going to construct

$$\Delta^{n+1} = (S'^{n+1}, \Sigma'^{n+1}, E'^{n+1}, \Sigma'^{n+1}_t, \mathrm{Def}'^{n+1})$$

first. Below, $\overline{S}^0 = S^0$ and $\overline{S}^n = S^n \setminus S^{n-1}$ if $n \geqslant 1$.

$$S'^{n+1} = S^n \cup \{(s,0), (s,1) \mid s \in \overline{S}^n\} \cong S^n \sqcup \overline{S}^n \sqcup \overline{S}^n,$$

$$\begin{aligned}
\Sigma'^{n+1}_t = \Sigma^n_t &\cup \{\tau^{(s,0)}\colon (s,0)^r \to (s,0) \mid r \geqslant 0, \tau \in \Sigma^{\mathcal{T}}_r, s \in \overline{S}^n\}\\
&\cup \{\tau^{(s,1)}\colon (s,1)^r \to (s,1) \mid r \geqslant 0, \tau \in \Sigma^{\mathcal{T}}_r, s \in \overline{S}^n\}\\
&\cup \{\delta^s\colon s^2 \to (s,0) \mid s \in \overline{S}^n\}\\
&\cup \{\omega^{(s,0)} \text{ constant operation symbol on } (s,0) \mid s \in \overline{S}^n\}\\
&\cup \{\eta^s, \varepsilon^s\colon (s,0) \times s \to (s,1) \mid s \in \overline{S}^n\},
\end{aligned}$$

$$\Sigma'^{n+1} = \Sigma^n \cup \Sigma_t'^{n+1} \cup \{\pi^s \colon (s,0) \times s \to s \mid s \in \overline{S}^n\},$$

$$\begin{aligned}
E'^{n+1} = E^n &\cup \{\delta^s(x,x) = \omega^{(s,0)} \mid s \in \overline{S}^n\} \\
&\cup \{\eta^s(\delta^s(x,y),y) = \varepsilon^s(\delta^s(x,y),y) \mid s \in \overline{S}^n\} \\
&\cup \{\pi^s(\delta^s(x,y),y) = x \mid s \in \overline{S}^n\} \\
&\cup \{\delta^s(\pi^s(x,y),y) = x \mid s \in \overline{S}^n\} \\
&\cup \{\text{all axioms from } \mathcal{T} \text{ for the } \tau^{(s,0)}\text{'s and the } \tau^{(s,1)}\text{'s} \mid s \in \overline{S}^n\} \\
&\cup \{\tau^{(s,0)}(\delta^s(x_1,y_1),\dots,\delta^s(x_r,y_r)) \\
&\quad = \delta^s(\tau^s(x_1,\dots,x_r),\tau^s(y_1,\dots,y_r)) \mid r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\} \\
&\cup \{\tau^{(s,0)}(\omega^{(s,0)},\dots,\omega^{(s,0)}) = \omega^{(s,0)} \mid r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\} \\
&\cup \{\tau^{(s,1)}(\eta^s(x_1,y_1),\dots,\eta^s(x_r,y_r)) \\
&\quad = \eta^s(\tau^{(s,0)}(x_1,\dots,x_r),\tau^s(y_1,\dots,y_r)) \mid r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\} \\
&\cup \{\tau^{(s,1)}(\varepsilon^s(x_1,y_1),\dots,\varepsilon^s(x_r,y_r)) \\
&\quad = \varepsilon^s(\tau^{(s,0)}(x_1,\dots,x_r),\tau^s(y_1,\dots,y_r)) \mid r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\} \\
&\cup \{\tau^s(\pi^s(x_1,y_1),\dots,\pi^s(x_r,y_r)) \\
&\quad = \pi^s(\tau^{(s,0)}(x_1,\dots,x_r),\tau^s(y_1,\dots,y_r)) \mid r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, s \in \overline{S}^n\}
\end{aligned}$$

and

$$\begin{cases} \mathrm{Def}'^{n+1}(\sigma) = \mathrm{Def}^n(\sigma) \text{ if } \sigma \in \Sigma^n \setminus \Sigma_t^n \\ \mathrm{Def}'^{n+1}(\pi^s) = \{\eta^s(x) = \varepsilon^s(x)\} \text{ for } s \in \overline{S}^n. \end{cases}$$

Hence, we have $\Gamma^n \subseteq \Delta^{n+1}$ and we consider the obvious $\mathcal{T}$-enrichment on $\mathrm{Mod}(\Delta^{n+1})$.

Let now T^{n+1} be the set of finitary terms $\theta \colon \prod_{i=1}^m s_i \to s$ of Σ'^{n+1} which are not terms of Σ'^n (where we consider $\Sigma'^0 = \varnothing$). We then define Γ^{n+1} as:

$$S^{n+1} = S'^{n+1} \cup \{s_\theta, s'_\theta \mid \theta \in T^{n+1}\} \cong S'^{n+1} \sqcup T^{n+1} \sqcup T^{n+1},$$

$$\begin{aligned}
\Sigma_t^{n+1} = \Sigma_t'^{n+1} &\cup \{\tau^{s_\theta} \colon s_\theta^r \to s_\theta \mid r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, \theta \in T^{n+1}\} \\
&\cup \{\tau^{s'_\theta} \colon (s'_\theta)^r \to s'_\theta \mid r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, \theta \in T^{n+1}\} \\
&\cup \{\alpha_\theta \colon s \to s_\theta \mid \theta \colon \prod_{i=1}^m s_i \to s \in T^{n+1}\} \\
&\cup \{\mu_\theta \colon \prod_{i=1}^m s_i \to s_\theta \mid \theta \colon \prod_{i=1}^m s_i \to s \in T^{n+1}\} \\
&\cup \{\eta_\theta, \varepsilon_\theta \colon s_\theta \to s'_\theta \mid \theta \in T^{n+1}\},
\end{aligned}$$

$$\Sigma^{n+1} = \Sigma'^{n+1} \cup \Sigma_t^{n+1} \cup \{\pi_\theta \colon s_\theta \to s \mid \theta \colon \prod_{i=1}^m s_i \to s \in T^{n+1}\},$$

$$
\begin{aligned}
E^{n+1} = E'^{n+1} &\cup \{\eta_\theta(\alpha_\theta(x)) = \varepsilon_\theta(\alpha_\theta(x)) \,|\, \theta \in T^{n+1}\} \\
&\cup \{\pi_\theta(\alpha_\theta(x)) = x \,|\, \theta \in T^{n+1}\} \\
&\cup \{\alpha_\theta(\pi_\theta(x)) = x \,|\, \theta \in T^{n+1}\} \\
&\cup \{\alpha_\theta(\theta(x_1,\dots,x_m)) = \mu_\theta(x_1,\dots,x_m) \,|\, \theta \colon \prod_{i=1}^m s_i \to s \in T^{n+1}\} \\
&\cup \{\text{all axioms from } \mathcal{T} \text{ for the } \tau^{s_\theta}\text{'s and the } \tau^{s'_\theta}\text{'s} \,|\, \theta \in T^{n+1}\} \\
&\cup \{\tau^{s_\theta}(\alpha_\theta(x_1),\dots,\alpha_\theta(x_r)) = \alpha_\theta(\tau^s(x_1,\dots,x_r)) \,| \\
&\qquad\qquad\qquad r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, \theta \colon \prod_{i=1}^m s_i \to s \in T^{n+1}\} \\
&\cup \{\tau^{s_\theta}(\mu_\theta(x_{11},\dots,x_{1m}),\dots,\mu_\theta(x_{r1},\dots,x_{rm})) \\
&\quad = \mu_\theta(\tau^{s_1}(x_{11},\dots,x_{r1}),\dots,\tau^{s_m}(x_{1m},\dots,x_{rm})) \,| \\
&\qquad\qquad\qquad r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, \theta \colon \prod_{i=1}^m s_i \to s \in T^{n+1}\} \\
&\cup \{\tau^{s'_\theta}(\eta_\theta(x_1),\dots,\eta_\theta(x_r)) = \eta_\theta(\tau^{s_\theta}(x_1,\dots,x_r)) \,| \\
&\qquad\qquad\qquad\qquad r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, \theta \in T^{n+1}\} \\
&\cup \{\tau^{s'_\theta}(\varepsilon_\theta(x_1),\dots,\varepsilon_\theta(x_r)) = \varepsilon_\theta(\tau^{s_\theta}(x_1,\dots,x_r)) \,| \\
&\qquad\qquad\qquad\qquad r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, \theta \in T^{n+1}\} \\
&\cup \{\tau^{s}(\pi_\theta(x_1),\dots,\pi_\theta(x_r)) = \pi_\theta(\tau^{s_\theta}(x_1,\dots,x_r)) \,| \\
&\qquad\qquad\qquad r \geqslant 0, \tau \in \Sigma_r^{\mathcal{T}}, \theta \colon \prod_{i=1}^m s_i \to s \in T^{n+1}\}
\end{aligned}
$$

and

$$
\begin{cases}
\mathrm{Def}^{n+1}(\sigma) = \mathrm{Def}'^{n+1}(\sigma) \text{ if } \sigma \in \Sigma'^{n+1} \setminus \Sigma_t'^{n+1} \\
\mathrm{Def}^{n+1}(\pi_\theta) = \{\eta_\theta(x) = \varepsilon_\theta(x)\} \text{ for } \theta \in T^{n+1}.
\end{cases}
$$

Thus, we have $\Delta^{n+1} \subseteq \Gamma^{n+1}$ and we consider the obvious $\mathcal{T}$-enrichment on $\mathrm{Mod}(\Gamma^{n+1})$. This ends the recursive definition of the series

$$\Gamma^0 \subseteq \Delta^1 \subseteq \Gamma^1 \subseteq \cdots$$

and we set $\Gamma^{\mathcal{T}}_{\text{proto}}$ to be the union of these finitary essentially algebraic theories. We provide $\mathrm{Mod}(\Gamma^{\mathcal{T}}_{\text{proto}})$ with the $\mathcal{T}$-enrichment coming from the $\mathcal{T}$-enrichments on the $\mathrm{Mod}(\Gamma^n)$'s. Since they will be the most important cases, we denote $\Gamma^{\mathrm{Th[Set]}}_{\text{proto}}$ simply by Γ_{proto} and $\Gamma^{\mathrm{Th[Set_*]}}_{\text{proto}}$ by Γ_{homo}.

Proposition 4.7. Let $\mathcal{T}$ be a commutative finitary one-sorted algebraic theory. The $\mathcal{T}$-enriched category $\mathrm{Mod}(\Gamma^{\mathcal{T}}_{\text{proto}})$ is regular and protomodular.

Proof. The 'Δ part' of the construction makes $\mathrm{Mod}(\Gamma^{\mathcal{T}}_{\text{proto}})$ protomodular. Indeed, considering $\pi^s \colon (s,0) \times s \to s$, $d_1 = \delta^s$ and $w_1 = \omega^{(s,0)}$, $\Gamma^{\mathcal{T}}_{\text{proto}}$ satisfies the conditions of Theorem 3.6.

On the other hand, the 'Γ ingredient' of the construction ensures that $\mathrm{Mod}(\Gamma^{\mathcal{T}}_{\text{proto}})$ is a regular category since each finitary term θ of $\Sigma^{\mathcal{T}}_{\text{proto}}$ is in T^{n+1} for some $n \geqslant 0$, which makes the conditions of Theorem 3.2 hold. ☐

Similarly to what we did for Janelidze matrix conditions, we still need some approximate co-operations. This has been treated in [11].

Theorem 4.8. (Theorem 6.4 in [11]) Let $\mathbb{C}$ be a finitely complete category with finite coproducts. Then $\mathbb{C}$ is protomodular if and only if, for each $X \in \mathbb{C}$, the morphism

$$\left(\begin{smallmatrix} d^X \\ \iota_2 \end{smallmatrix}\right) : D(X) + X \to X + X$$

is a strong epimorphism where the square

$$\begin{tikzcd} D(X) \arrow[r, "d^X"] \arrow[d, "e^X"'] \arrow[dr, phantom, "\lrcorner", very near start] & X + X \arrow[d, "\left(\begin{smallmatrix} 1_X \\ 1_X \end{smallmatrix}\right)"] \\ 0 \arrow[r, "!"'] & X \end{tikzcd}$$

is a pullback, 0 the initial object and $\iota_2 \colon X \to X + X$ is the second coproduct injection.

Finally, we also need to prove that, under some assumptions, regular protomodularity is also a property which lifts from $\mathbb{C}$ to $\widetilde{\mathbb{C}}$. This is where colimits are needed.

Proposition 4.9. Let $\mathbb{C}$ be a small regular protomodular category. Suppose either that $\mathbb{C}$ has binary coproducts or that $\mathbb{C}$ has pushouts of split monomorphisms along arbitrary morphisms. Then $\widetilde{\mathbb{C}}$ is also a regular protomodular category.

Proof. This has been (or will be) proved in [22, 24] using the theory of unconditional exactness properties. A direct proof also exists using a similar technique to what we did in Proposition 4.4. It uses here the fact that a finitely complete category is protomodular if and only if, for each morphism of points $(u, v) \colon (p, s) \to (p', s')$ for which the square $p'u = vp$ is a pullback,

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, shift right, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, shift right, "p'"'] \\ B \arrow[r, "v"'] \arrow[u, shift right, "s"'] & B' \arrow[u, shift right, "s'"'] \end{tikzcd}$$

the morphisms u and s' are jointly strongly epimorphic. If binary coproducts exist and in a regular context, this means that the factorisation $\left(\begin{smallmatrix} u \\ s' \end{smallmatrix}\right) : A + B' \to A'$ is a regular epimorphism. If the pushout of s along v exists and in a regular context, this means that the factorisation of the upward commutative square above through this pushout is a regular epimorphism. In both cases, a similar technique than in Proposition 4.4 can be applied. $\square$

Putting together all these ingredients, we have an embedding theorem for regular protomodular and homological categories (assuming the existence of some colimits). The proof is again similar to the one in [23] and we omit it. The small differences for the protomodular case can be found in [22].

Theorem 4.10. Let $\mathcal{T}$ be a commutative finitary one-sorted algebraic theory and $\mathbb{C}$ a small regular protomodular $\mathcal{T}$-enriched category. Suppose also either that $\mathbb{C}$ has binary coproducts or that $\mathbb{C}$ has pushouts of split monomorphisms along arbitrary morphisms. We denote by $\mathrm{Sub}(1)$ the set of subobjects of the terminal object 1 of $\mathbb{C}$. Then, there exists a faithful $\mathcal{T}$-enriched embedding $\phi \colon \mathbb{C} \hookrightarrow \mathrm{Mod}(\Gamma^{\mathcal{T}}_{\mathrm{proto}})^{\mathrm{Sub}(1)}$ which preserves and reflects finite limits, isomorphisms and regular epimorphisms. Moreover, for each morphism $f \colon C \to C'$ in $\mathbb{C}$, each $I \in \mathrm{Sub}(1)$ and each $s \in S^{\mathcal{T}}_{\mathrm{proto}}$,

$$(\mathrm{Im}\, \phi(f)_I)_s = \{(\phi(f)_I)_s(x) \mid x \in (\phi(C)_I)_s\}.$$

Remark 4.11. The assumption about colimits in Theorem 4.10 is only used to prove that $\mathrm{Lex}(\mathbb{C},\mathrm{Set})^{\mathrm{op}}$ is also protomodular. If one has another condition on the small regular protomodular category $\mathbb{C}$ which also implies that $\widetilde{\mathbb{C}}$ is protomodular, such an embedding will also exist.

In the pointed context, $\mathrm{Sub}(1)$ is reduced to a singleton. We thus have the following corollary.

Corollary 4.12. Let $\mathbb{C}$ be a small homological category with binary coproducts or pushouts of split monomorphisms along arbitrary morphisms. Then, there exists a regular conservative embedding $\mathbb{C} \hookrightarrow \mathrm{Mod}(\Gamma_{\mathrm{homo}})$.

Proof. Let $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$ in Theorem 4.10. □

5 Applications

Analogously to the metatheorems of [5], our Embedding Theorem 4.6 for Janelidze matrix conditions gives a way to prove some statements in regular $\mathcal{T}$-enriched categories with (M, X)-closed relations in an 'essentially algebraic way' as follows. Besides, our Embedding Theorem 4.10 for protomodular categories cannot be used in that way, unless the existence of coproducts or pushouts is assumed.

Let (M, X) be an extended matrix of terms in the commutative finitary one-sorted algebraic theory $\mathcal{T}$. Consider a statement P of the form $\psi \Rightarrow \omega$, where ψ and ω are conjunctions of properties which can be expressed as

1. some finite diagram is commutative,
2. some finite diagram is a limit diagram,
3. the equality $\tau(f_1, \ldots, f_r) = g$ holds for an r-ary term τ of $\mathcal{T}$ and parallel morphisms $f_1, \ldots, f_r, g$,
4. some morphism is a monomorphism,
5. some morphism is a regular epimorphism,
6. some morphism is an isomorphism,
7. some morphism factors through a given monomorphism.

Then, this statement P is valid in all regular $\mathcal{T}$-enriched $\mathcal{V}$-categories with (M, X)-closed relations (for all universes $\mathcal{V}$) if and only if it is valid in $\mathrm{Mod}(\Gamma_{(M,X)})$ (for all universes). Indeed, in view of Proposition 4.1, the 'only if part' is obvious. Conversely, if $\mathbb{C}$ is a regular $\mathcal{T}$-enriched category with (M, X)-closed relations, we can suppose it is small up to a change of universe. Therefore, by Theorem 4.6, it suffices to prove P in $\mathrm{Mod}(\Gamma_{(M,X)})^{\mathrm{Sub}(1)}$. Since every part of the statement P is 'componentwise', it is enough to prove it in $\mathrm{Mod}(\Gamma_{(M,X)})$.

At a first glance, one could think this technique will be hard to use in practice, in view of the difficult definition of $\mathrm{Mod}(\Gamma_{(M,X)})$. However, due to the additional property in our Theorem 4.6, we can suppose the homomorphisms $f\colon A \to B$ considered in the statement P have an easy description of their images, i.e.,

$$(\mathrm{Im}\, f)_s = \{f_s(a) \mid a \in A_s\}$$

for all $s \in S_{(M,X)}$. In particular, if f is a regular epimorphism, f_s will be a surjective function for all $s \in S_{(M,X)}$. Therefore, in practice, it seems we will never have to use the operations α_θ, μ_θ, η_θ, ε_θ and π_θ. They were built only to make $\mathrm{Mod}(\Gamma_{(M,X)})$ a regular category.

We now show on concrete examples how to use the embedding theorem and prove some results using elements and operations. A first example can be found in [23] in the Mal'tsev case and a second one in [25] in the n-permutable case. Let us give here one in the subtractive case and one in the Goursat case. We recall from our Example 2.3 that a pointed (i.e., Th[Set$_*$]-enriched) regular category is subtractive if and only if it has $\left(\left(\begin{array}{cc|c} x & 0 & x \\ x & x & 0 \end{array}\right), \emptyset\right)$-closed relations.

Lemma 5.1. [12] Let $\mathbb{C}$ be a regular subtractive category and d an approximate subtraction in $\mathbb{C}$ (i.e., a morphism $d\colon A \times A \to B$ such that $d(1,1) = 0$).

$$\begin{tikzcd} A \arrow[d, "{(1,1)}"'] \arrow[dr, "0"] & \\ A \times A \arrow[r, "d"'] & B \end{tikzcd}$$

Let also $x, y, z, w\colon C \to A$ be four morphisms in $\mathbb{C}$ such that $d(x,y) = d(z,t)$. Then $d(x,z) = d(y,t)$.

Proof. By our embedding Theorem 4.6, it is enough to prove this lemma in $\mathrm{Mod}(\Gamma_{(M,X)})$ with $(M,X) = \left(\left(\begin{array}{cc|c} x & 0 & x \\ x & x & 0 \end{array}\right), \emptyset\right)$. So, let $s \in S_{(M,X)}$ and $c \in C_s$. We can compute:

$$\begin{aligned} \alpha^s(d(x(c), z(c))) &= \rho_1^s(d(x(c), z(c)), 0^s) \\ &= \rho_1^s(d(x(c), z(c)), d(z(c), z(c))) \\ &= d(\rho_1^s((x(c), z(c)), (z(c), z(c)))) \\ &= d(\rho_1^s(x(c), z(c)), \rho_1^s(z(c), z(c))) \\ &= d(\rho_1^s(x(c), z(c)), 0^{(s,0)}) \\ &= d(\rho_1^s(x(c), z(c)), \rho_1^s(y(c), y(c))) \\ &= d(\rho_1^s((x(c), y(c)), (z(c), y(c)))) \\ &= \rho_1^s(d(x(c), y(c)), d(z(c), y(c))) \\ &= \rho_1^s(d(z(c), t(c)), d(z(c), y(c))) \\ &= d(\rho_1^s(z(c), z(c)), \rho_1^s(t(c), y(c))) \\ &= d(\rho_1^s(y(c), y(c)), \rho_1^s(t(c), y(c))) \\ &= \rho_1^s(d(y(c), t(c)), d(y(c), y(c))) \\ &= \rho_1^s(d(y(c), t(c)), 0^s) \\ &= \alpha^s(d(y(c), t(c))). \end{aligned}$$

Since $\pi^s(\alpha^s(x))$ is everywhere-defined and $\pi^s(\alpha^s(x)) = x$ holds for each $x \in B_s$, $\alpha^s\colon B_s \to B_{(s,0)}$ is injective. We can thus deduce from the above calculation that $d(x(c), z(c)) = d(y(c), t(c))$, which concludes the proof. $\square$

Notice that this proof is directly inspired from the proof of the same lemma in subtractive varieties. For a general extended matrix (M,X), this will often be the case. Indeed, many (but not all) proofs for $\mathcal{T}$-enriched algebraic varieties with (M,X)-closed relations can be transposed into a proof in $\mathrm{Mod}(\Gamma_{(M,X)})$.

The same phenomenon appears again in the next example, in the 3-permutable context. Example 2.3 tells us that a regular category is Goursat (i.e., 3-permutable) if and only if it has $\left(\left(\begin{array}{ccc|cc} x & y & y & x & z \\ x & x & y & z & y \end{array}\right), \{z\}\right)$-closed relations. Next lemma admits a generalisation for n-permutable categories, but we set $n=3$ in order to keep computations simple. Let us recall that in a regular category, the composition of the relations $(r_1,r_2)\colon R \rightarrowtail A\times B$ (represented as $R\colon A \to B$) and $(s_1,s_2)\colon S \rightarrowtail B\times C$ (represented as $S\colon B\to C$) is denoted $S\circ R = SR\colon A \to C$ and is defined as $(t_1,t_2)\colon S\circ R \rightarrowtail A\times C$ where $(t_1,t_2)q$ is the image factorisation of $(r_1p_1, s_2p_2)\colon P \to A\times C$ and (p_1,p_2) is the pullback of s_1 along r_2.

$$\begin{tikzcd}
 & & P \arrow[dl, "p_1"'] \arrow[d, two heads, "q"] \arrow[dr, "p_2"] & & \\
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & S\circ R \arrow[dll, "t_1"] \arrow[drr, "t_2"'] & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\
A & & B & & C
\end{tikzcd}$$

We also denote by R° the opposite relation $(r_2,r_1)\colon R \rightarrowtail B\times A$.

Lemma 5.2. (Theorem 3.5 in [13]) Let $R \rightarrowtail A\times B$ be a relation in a regular Goursat category $\mathbb{C}$. Then, $R^\circ R R^\circ R \leqslant R^\circ R$.

Proof. By our embedding Theorem 4.6, it is enough to prove this lemma in $\mathrm{Mod}(\Gamma_{(M,X)})$ with $(M,X) = \left(\left(\begin{array}{ccc|cc} x & y & y & x & z \\ x & x & y & z & y \end{array}\right), \{z\}\right)$. Let $s\in S_{(M,X)}$. Considering the description of images of morphisms on the form $\phi(f)_I$ in Theorem 4.6 and the above definition of $R^\circ R$, we can assume without loss of generality that

$$(R^\circ R)_s = \{(a,a')\in A_s\times A_s \mid \exists b\in B_s \text{ such that } aRb, a'Rb\}.$$

In the same way, we can assume without loss of generality that

$$(R^\circ R R^\circ R)_s = \{(a,a'')\in A_s\times A_s \mid \exists a'\in A_s, b,b'\in B_s \text{ such that } aRb, a'Rb, a'Rb', a''Rb'\}.$$

So, let $(a,a'')\in (R^\circ R R^\circ R)_s$ and we want to prove that $(a,a'')\in (R^\circ R)_s$. Since $(a,b)\in R_s$, $(a',b)\in R_s$ and $(a',b')\in R_s$ (for some a',b,b'), we know that

$$(\rho_1^s(a,a',a'), \rho_1^s(b,b,b')) = (\alpha^s(a), \kappa_1^s(b,b'))\in R_{(s,0)}.$$

Moreover, since $(a',b)\in R_s$, $(a',b')\in R_s$ and $(a'',b')\in R_s$, we also know that

$$(\rho_2^s(a',a',a''), \rho_2^s(b,b',b')) = (\alpha^s(a''), \kappa_1^s(b,b'))\in R_{(s,0)}.$$

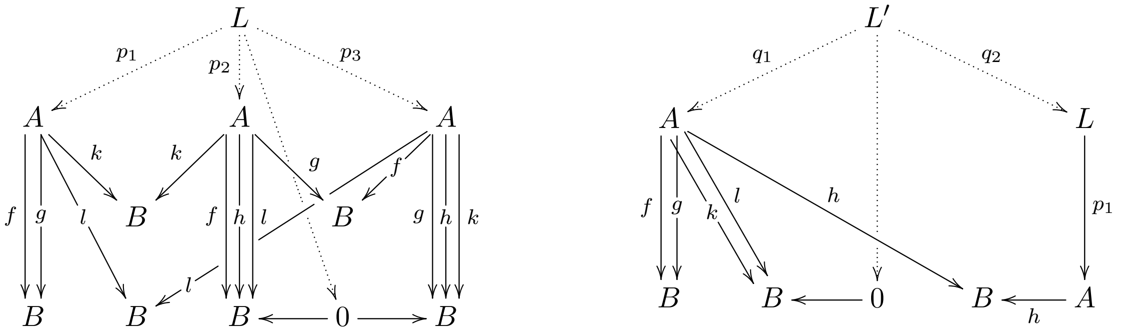
Therefore, $\alpha^s(a,a'') = (\alpha^s(a),\alpha^s(a''))\in (R^\circ R)_{(s,0)}$ and $(a,a'') = \pi^s(\alpha^s(a,a''))\in (R^\circ R)_s$. $\square$

We warn the reader here that all algebraic proofs cannot be rewritten as they are in an essentially algebraic way (and in particular in $\mathrm{Mod}(\Gamma_{(M,X)})$). For instance, let us consider the following 'ad hoc' property (P_{sub}) of a pointed regular category:

Given five parallel morphisms

$$\begin{tikzcd} A \arrow[r, shift left=4, "f"] \arrow[r, shift left=2, "g" description] \arrow[r, "h" description] \arrow[r, shift right=2, "k" description] \arrow[r, shift right=4, "l"'] & B, \end{tikzcd}$$

we consider the limits L and L' of the diagrams made of plain arrows below:



Then (P_{sub}) requires that q_2 is a regular epimorphism.

Lemma 5.3. Let $\mathcal{K}$ be a finitary one-sorted algebraic theory such that $\mathrm{Alg}_{\mathcal{K}}$ is a subtractive category. Then $\mathrm{Alg}_{\mathcal{K}}$ satisfies the property (P_{sub}).

Proof. From [27] or from Theorem 2.2, we know there exists a binary term d in $\mathcal{K}$ such that

$$\begin{cases} d(x,0) = x \\ d(x,x) = 0 \end{cases}$$

are theorems in $\mathcal{K}$ (where 0 is the unique constant of $\mathcal{K}$). The property (P_{sub}) means that for each triple $(x,y,z) \in A^3$ satisfying the identities

$$\begin{cases} f(x) = g(x) \\ k(x) = k(y) \\ f(y) = h(y) = l(y) = 0 \end{cases} \qquad \begin{cases} g(y) = f(z) \\ g(z) = h(z) = k(z) = 0 \\ l(x) = l(z), \end{cases}$$

there exists an element $a \in A$ such that $f(a) = g(a)$, $k(a) = l(a) = 0$ and $h(a) = h(x)$. In the subtractive variety $\mathrm{Alg}_{\mathcal{K}}$, it suffices to consider $a = d(d(x,y),z)$. Indeed, we have

$$\begin{aligned} f(a) &= d(d(f(x),f(y)),f(z)) \\ &= d(d(g(x),0),g(y)) \\ &= d(g(x),g(y)) \\ &= d(d(g(x),g(y)),0) \\ &= d(d(g(x),g(y)),g(z)) \\ &= g(a) \end{aligned}$$

$$k(a) = d(d(k(x),k(y)),k(z)) = d(d(k(x),k(x)),0) = d(k(x),k(x)) = 0$$
$$l(a) = d(d(l(x),l(y)),l(z)) = d(d(l(x),0),l(x)) = d(l(x),l(x)) = 0$$

and

$$h(a) = d(d(h(x),h(y)),h(z)) = d(d(h(x),0),0) = d(h(x),0) = h(x). \qquad \square$$

Although this property (P_{sub}) holds for subtractive varieties, it seems it is not true for all regular subtractive categories. In particular, the direct translation of the above proof in $\mathrm{Mod}(\Gamma_{(M,X)})$ for $(M,X) = \left(\left(\begin{array}{cc|c} x & 0 & x \\ x & x & 0 \end{array} \right), \emptyset \right)$ does not work. The reason is that the equality

$$d(d(g(x),0),g(y)) = d(g(x),g(y)) = d(d(g(x),g(y)),0)$$

should become

$$\begin{aligned} \rho_1^{(s,0)}(\rho_1^s(g(x),0^s),\alpha^s(g(y))) &= \rho_1^{(s,0)}(\alpha^s(g(x)),\alpha^s(g(y))) \\ &\overset{(*)}{=} \alpha^{(s,0)}(\rho_1^s(g(x),g(y))) \\ &= \rho_1^{(s,0)}(\rho_1^s(g(x),g(y)),0^{(s,0)}) \end{aligned}$$

where the equality $(*)$ does not seem to hold. The morphisms h, k and l from property (P_{sub}) where constructed only to prevent the existence of another easy proof of the same property in $\mathrm{Mod}(\Gamma_{(M,X)})$.

Intuitively, this counter-example tells us that the order in which we apply the operations matters. In the algebraic proof above, the term with d applied non-trivially first (i.e., $d(d(g(x),g(y)),0)$) was equal to the term with d applied non-trivially in second position (i.e., $d(d(g(x),0),g(y))$). As we can see this kind of equality seems to be impossible to transpose in $\mathrm{Mod}(\Gamma_{(M,X)})$.

6 The exact context

We recall that a regular category $\mathbb{C}$ is *exact* (in the sense of Barr [3]) if every equivalence relation is a kernel pair. The aim of this section is to prove similar results than Theorems 4.6 and 4.10 for exact $\mathcal{T}$-enriched categories with (M,X)-closed relations and for exact $\mathcal{T}$-enriched protomodular categories. Now, the 'representing categories' are $\mathrm{Mod}(\Gamma_{(M,X)})_{\mathrm{ex}/\mathrm{reg}}$ and $\mathrm{Mod}(\Gamma^{\mathcal{T}}_{\mathrm{proto}})_{\mathrm{ex}/\mathrm{reg}}$, the exact completions of $\mathrm{Mod}(\Gamma_{(M,X)})$ and $\mathrm{Mod}(\Gamma^{\mathcal{T}}_{\mathrm{proto}})$ respectively. We recall here the exact completion of a regular category, which first appeared in [33].

Let $\mathbb{C}$ be a well-powered regular category. Its exact completion $\mathbb{C}_{\mathrm{ex}/\mathrm{reg}}$ is defined as the following category:

- objects of $\mathbb{C}_{\mathrm{ex}/\mathrm{reg}}$ are pairs (A,R) where A is an object of $\mathbb{C}$ and R an equivalence relation on A.
- arrows $\alpha\colon (A,R) \to (B,S)$ are relations $\alpha\colon A \to B$ such that
  1. $S\alpha R = \alpha$
  2. $\alpha\alpha^\circ \leqslant S$
  3. $R \leqslant \alpha^\circ\alpha$
- The identity on (A,R) is $R\colon A \to A$.
- Composition is the composition of relations.

We then get a functor

$$\begin{aligned} I\colon \mathbb{C} &\longrightarrow \mathbb{C}_{\mathrm{ex}/\mathrm{reg}} \\ A &\longmapsto (A,\Delta_A) \\ f\colon A \to B &\longmapsto (1_A,f)\colon A \rightarrowtail A\times B \end{aligned}$$

where Δ_A is the discrete equivalence relation on A, i.e., $(1_A, 1_A)\colon A \rightarrowtail A \times A$.

Proposition 6.1. [33] Let $\mathbb{C}$ be a well-powered regular category. Then, $\mathbb{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$ is exact, $I\colon \mathbb{C} \hookrightarrow \mathbb{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$ is fully faithful and preserves finite limits and regular epimorphisms. It is the exact completion of $\mathbb{C}$ in the sense that, for each functor $F\colon \mathbb{C} \to \mathbb{D}$ which preserves finite limits and regular epimorphisms for an exact category $\mathbb{D}$, there exists a unique (up to isomorphism) functor $\widehat{F}\colon \mathbb{C}_{\mathrm{ex}\,/\,\mathrm{reg}} \to \mathbb{D}$ which preserves finite limits and regular epimorphisms and such that $\widehat{F}I = F$.

$$\begin{tikzcd} \mathbb{C} \arrow[rr, hook, "I"] \arrow[dr, "F"'] & & \mathbb{C}_{\mathrm{ex}\,/\,\mathrm{reg}} \arrow[dl, dotted, "\widehat{F}"] \\ & \mathbb{D} & \end{tikzcd}$$

Now, if we consider a $\mathcal{T}$-enrichment on $\mathbb{C}$ for a finitary one-sorted algebraic theory $\mathcal{T}$, we can build one on $\mathbb{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$. Indeed, for each r-ary term τ of $\mathcal{T}$ and each object (A, R) of $\mathbb{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$, we consider the map

$$(A, R)^r = (A^r, R^r) \xrightarrow{R\circ(1_{A^r}, \tau^A)} (A, R)$$

where R^r denotes here the equivalence relation given by the product

$$R \times \cdots \times R \rightarrowtail A^2 \times \cdots \times A^2 \cong A^r \times A^r.$$

It is easy to see that this defines a $\mathcal{T}$-enrichment on $\mathbb{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$ such that $i\colon \mathbb{C} \hookrightarrow \mathbb{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$ is a $\mathcal{T}$-enriched functor. Moreover, this makes $\mathbb{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$ the $\mathcal{T}$-enriched exact completion of $\mathbb{C}$, in the sense that, with the notation of Proposition 6.1, if $\mathbb{D}$ and F are $\mathcal{T}$-enriched, $\widehat{F}$ is also $\mathcal{T}$-enriched. We now need a few results in order to get our embedding theorem.

Lemma 6.2. Let (M, X) be an extended matrix of terms in the finitary one-sorted algebraic theory $\mathcal{T}$ as in (1). Let also $r\colon R \rightarrowtail A^a$ be an a-ary relation in the regular $\mathcal{T}$-enriched category $\mathbb{C}$. If $p\colon B \twoheadrightarrow A$ is a regular epimorphism and if we consider the pullback

$$\begin{tikzcd} S \arrow[r, two heads, "q"] \arrow[d, tail, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, tail, "r"] \\ B^a \arrow[r, two heads, "p^a"'] & A^a \end{tikzcd}$$

then, r is (M, X)-closed if and only if s is (M, X)-closed.

Proof. We are going to use Proposition 2.1. Let us suppose firstly that r is (M, X)-closed and let $(y_1, \ldots, y_l)\colon Y \to B^l$ be such that

$$(t_{1j}(y_1, \ldots, y_l), \ldots, t_{aj}(y_1, \ldots, y_l)) = sv_j\colon Y \to B^a$$

for some $v_1, \ldots, v_b\colon Y \to S$. Thus,

$$\begin{aligned}(t_{1j}(py_1, \ldots, py_l), \ldots, t_{aj}(py_1, \ldots, py_l)) &= p^a(t_{1j}(y_1, \ldots, y_l), \ldots, t_{aj}(y_1, \ldots, y_l)) \\ &= p^a s v_j \\ &= rqv_j\end{aligned}$$

factors through r. Since r is (M, X)-closed, there is a regular epimorphism $p'\colon Z \twoheadrightarrow Y$ and morphisms $z_{l+1}, \dots, z_k \colon Z \to A$ such that

$$(u_{1j}(py_1p', \dots, py_lp', z_{l+1}, \dots, z_k), \dots, u_{aj}(py_1p', \dots, py_lp', z_{l+1}, \dots, z_k)) = rw_j$$

for some $w_1, \dots, w_{b'} \colon Z \to R$. Now, we consider the pullback

$$\begin{tikzcd}
Z' \arrow[r, "(z'_{l+1},\dots,z'_k)"] \arrow[d, "q'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B^{k-l} \arrow[d, "p^{k-l}"] \\
Z \arrow[r, "(z_{l+1},\dots,z_k)"'] & A^{k-l}
\end{tikzcd}$$

and we prove that the required property is satisfied with the regular epimorphism $p'q' : Z' \twoheadrightarrow Y$ and the morphisms $z'_{l+1}, \dots, z'_k \colon Z' \to B$. In view of the definition of s, we only have to notice that

$$\begin{aligned}
&p^a(u_{1j}(y_1p'q', \dots, y_lp'q', z'_{l+1}, \dots, z'_k), \dots, u_{aj}(y_1p'q', \dots, y_lp'q', z'_{l+1}, \dots, z'_k)) \\
&= (u_{1j}(py_1p'q', \dots, py_lp'q', z_{l+1}q', \dots, z_kq'), \dots, u_{aj}(py_1p'q', \dots, py_lp'q', z_{l+1}q', \dots, z_kq')) \\
&= rw_jq'
\end{aligned}$$

factors through r for all $j \in \{1, \dots, b'\}$.

Conversely, let us suppose s is (M, X)-closed and consider a morphism $(y_1, \dots, y_l) : Y \to A^l$ such that

$$(t_{1j}(y_1, \dots, y_l), \dots, t_{aj}(y_1, \dots, y_l)) = rv_j \colon Y \to A^a$$

for some $v_1, \dots, v_b \colon Y \to R$. We also consider the pullback

$$\begin{tikzcd}
Y' \arrow[r, "(y'_1,\dots,y'_l)"] \arrow[d, "q'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B^{l} \arrow[d, "p^{l}"] \\
Y \arrow[r, "(y_1,\dots,y_l)"'] & A^{l}.
\end{tikzcd}$$

Since

$$\begin{aligned}
p^a(t_{1j}(y'_1, \dots, y'_l), \dots, t_{aj}(y'_1, \dots, y'_l)) &= (t_{1j}(y_1q', \dots, y_lq'), \dots, t_{aj}(y_1q', \dots, y_lq')) \\
&= rv_jq',
\end{aligned}$$

the morphism

$$(t_{1j}(y'_1, \dots, y'_l), \dots, t_{aj}(y'_1, \dots, y'_l)) \colon Y' \to B^a$$

factors through s for all $j \in \{1, \dots, b\}$. But s is (M, X)-closed, so there exists a regular epimorphism $p'\colon Z \twoheadrightarrow Y'$ and morphisms $z_{l+1}, \dots, z_k \colon Z \to B$ such that

$$(u_{1j}(y'_1p', \dots, y'_lp', z_{l+1}, \dots, z_k), \dots, u_{aj}(y'_1p', \dots, y'_lp', z_{l+1}, \dots, z_k)) = sw_j$$

for some $w_1, \dots, w_{b'} \colon Z \to S$. Now, the required property is satisfied with the regular epimorphism $q'p' \colon Z \twoheadrightarrow Y$ and the morphisms $pz_{l+1}, \dots, pz_k \colon Z \to A$. Indeed,

$$\begin{aligned}
&(u_{1j}(y_1q'p', \dots, y_lq'p', pz_{l+1}, \dots, pz_k), \dots, u_{aj}(y_1q'p', \dots, y_lq'p', pz_{l+1}, \dots, pz_k)) \\
&= p^a(u_{1j}(y'_1p', \dots, y'_lp', z_{l+1}, \dots, z_k), \dots, u_{aj}(y'_1p', \dots, y'_lp', z_{l+1}, \dots, z_k)) \\
&= p^a sw_j \\
&= rqw_j
\end{aligned}$$

factors through r for all $j \in \{1, \dots, b'\}$. $\square$

Lemma 6.3. Let (M, X) be an extended matrix of terms in the finitary one-sorted algebraic theory $\mathcal{T}$ as in (1). Let also $r\colon R \rightarrowtail A^a$ be an a-ary relation in the well-powered regular $\mathcal{T}$-enriched category $\mathbb{C}$. This gives an a-ary relation $I(r)\colon I(R) \rightarrowtail I(A^a) \cong I(A)^a$ in $\mathbb{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$. Then, r is (M, X)-closed if and only if $I(r)$ is (M, X)-closed.

Proof. This comes from the fact that $I\colon \mathbb{C} \hookrightarrow \mathbb{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$ is $\mathcal{T}$-enriched and preserves and reflects finite limits and regular epimorphisms. $\square$

It is proved in [19] that if $\mathbb{C}$ is a regular well-powered Mal'tsev category, then its exact completion $\mathbb{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$ is also a Mal'tsev category. We now generalise this result for Janelidze matrix conditions.

Theorem 6.4. Let $n \geqslant 0$ and $(M_1, X_1), \ldots, (M_n, X_n)$ and (M, X) be extended matrices of terms in the finitary one-sorted algebraic theory $\mathcal{T}$ (with the same number a of lines). Let also $\mathbb{C}$ be a well-powered regular $\mathcal{T}$-enriched category. If every a-ary relation of $\mathbb{C}$ which is (M_i, X_i)-closed for all $i \in \{1, \ldots, n\}$ is also (M, X)-closed, then the same occurs in $\mathbb{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$.

Proof. Let $r\colon R \rightarrowtail A^a$ be an a-ary relation of $\mathbb{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$ which is (M_i, X_i)-closed for all $i \in \{1, \ldots, n\}$. It is proved in [37] that there exists an object $B \in \mathbb{C}$ and a regular epimorphism $p\colon I(B) \twoheadrightarrow A$ in $\mathbb{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$. So, we can consider the pullback

$$\begin{tikzcd}
\bullet \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & R \arrow[d, tail, "r"] \\
I(B)^a \arrow[r, two heads, "p^a"'] & A^a.
\end{tikzcd}$$

Moreover, it is shown in [37], that under the embedding $I\colon \mathbb{C} \hookrightarrow \mathbb{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$, $\mathbb{C}$ is closed under subobjects in $\mathbb{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$ (up to isomorphisms). Hence, we have the following pullback in $\mathbb{C}_{\mathrm{ex}\,/\,\mathrm{reg}}$

$$\begin{tikzcd}
I(S) \arrow[r, two heads, "q"] \arrow[d, tail, "I(s)"'] \arrow[dr, phantom, very near start, "\lrcorner"] & R \arrow[d, tail, "r"] \\
I(B^a) \arrow[r, two heads, "p^a"'] & A^a
\end{tikzcd}$$

for some a-ary relation $s\colon S \rightarrowtail B^a$ in $\mathbb{C}$. Now, by Lemma 6.2, $I(s)$ is (M_i, X_i)-closed for all $i \in \{1, \ldots, n\}$. By Lemma 6.3, s is (M_i, X_i)-closed for all $i \in \{1, \ldots, n\}$. Thus, by the assumption on $\mathbb{C}$, s is (M, X)-closed. Again by Lemma 6.3, $I(s)$ is (M, X)-closed and by Lemma 6.2, r is (M, X)-closed. $\square$

Corollary 6.5. Let $\mathcal{T}$ be a commutative finitary one-sorted algebraic theory and (M, X) an extended matrix of terms in $\mathcal{T}$. Then, $\mathrm{Mod}(\Gamma_{(M,X)})_{\mathrm{ex}\,/\,\mathrm{reg}}$ is exact with (M, X)-closed relations.

Proof. It is standard to prove that $\mathrm{Mod}(\Gamma_{(M,X)})$ is well-powered. Then $\mathrm{Mod}(\Gamma_{(M,X)})_{\mathrm{ex}\,/\,\mathrm{reg}}$ is exact from Proposition 6.1 and has (M, X)-closed relations by Propositions 4.1 and 6.4. $\square$

Proposition 6.6. Let $\mathcal{T}$ be a commutative finitary one-sorted algebraic theory. Then, $\mathrm{Mod}(\Gamma^{\mathcal{T}}_{\mathrm{proto}})_{\mathrm{ex}\,/\,\mathrm{reg}}$ is an exact $\mathcal{T}$-enriched protomodular category.

Proof. Again, $\mathrm{Mod}(\Gamma^{\mathcal{T}}_{\mathrm{proto}})$ is well-powered and regular, so $\mathrm{Mod}(\Gamma^{\mathcal{T}}_{\mathrm{proto}})_{\mathrm{ex/reg}}$ is an exact $\mathcal{T}$-enriched category. To conclude, we use Proposition 5.2 in [19] which says that the exact completion of a well-powered regular protomodular category is also protomodular. $\square$

In the pointed context (i.e., $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$), this also follows from Theorem 6.4 and from the fact [29] that a finitely complete pointed category is protomodular if and only if each binary relation which is $\left(\left(\begin{array}{c|c} x & x \\ y & x \end{array}\right), \varnothing\right)$-closed and $\left(\left(\begin{array}{c|c} 0 & y \\ y & 0 \end{array}\right), \varnothing\right)$-closed is also $\left(\left(\begin{array}{c|c} x & y \\ y & x \end{array}\right), \varnothing\right)$-closed.

With this in mind, we can state our embedding theorems in the exact context.

Theorem 6.7. Let $\mathcal{T}$ be a commutative finitary one-sorted algebraic theory, (M,X) an extended matrix of terms in $\mathcal{T}$ and $\mathbb{C}$ a small exact $\mathcal{T}$-enriched category with (M,X)-closed relations. We denote by $\mathrm{Sub}(1)$ the set of subobjects of the terminal object 1 of $\mathbb{C}$. Then, there exists a faithful $\mathcal{T}$-enriched embedding $\phi\colon \mathbb{C} \hookrightarrow (\mathrm{Mod}(\Gamma_{(M,X)})_{\mathrm{ex/reg}})^{\mathrm{Sub}(1)}$ which preserves and reflects finite limits, isomorphisms and coequalisers of equivalence relations.

Proof. We just have to compose the embedding of Theorem 4.6 with the embedding $I^{\mathrm{Sub}(1)}\colon \mathrm{Mod}(\Gamma_{(M,X)})^{\mathrm{Sub}(1)} \hookrightarrow (\mathrm{Mod}(\Gamma_{(M,X)})_{\mathrm{ex/reg}})^{\mathrm{Sub}(1)}$. $\square$

Remark 6.8. Theorem 6.7 is stated in such a way that it characterises small exact categories with (M,X)-closed relations among all small $\mathcal{T}$-enriched categories with finite limits and coequalisers of equivalence relations. In an analogous way, Theorem 4.6 characterises small regular categories with (M,X)-closed relations among all small $\mathcal{T}$-enriched categories with finite limits and coequalisers of kernel pairs.

Theorem 6.9. Let $\mathcal{T}$ be a commutative finitary one-sorted algebraic theory and $\mathbb{C}$ a small exact $\mathcal{T}$-enriched protomodular category. Suppose also either that $\mathbb{C}$ has binary coproducts or that $\mathbb{C}$ has pushouts of split monomorphisms along arbitrary morphisms. We denote by $\mathrm{Sub}(1)$ the set of subobjects of the terminal object 1 of $\mathbb{C}$. Then, there exists a faithful $\mathcal{T}$-enriched embedding $\phi\colon \mathbb{C} \hookrightarrow (\mathrm{Mod}(\Gamma^{\mathcal{T}}_{\mathrm{proto}})_{\mathrm{ex/reg}})^{\mathrm{Sub}(1)}$ which preserves and reflects finite limits, isomorphisms and coequalisers of equivalence relations.

Proof. We just have to compose the embedding of Theorem 4.10 with the embedding $I^{\mathrm{Sub}(1)}\colon \mathrm{Mod}(\Gamma^{\mathcal{T}}_{\mathrm{proto}})^{\mathrm{Sub}(1)} \hookrightarrow (\mathrm{Mod}(\Gamma^{\mathcal{T}}_{\mathrm{proto}})_{\mathrm{ex/reg}})^{\mathrm{Sub}(1)}$. $\square$

Let us particularise this result in the case $\mathcal{T} = \mathrm{Th}[\mathrm{Set}_*]$. We recall from [26] that a *semi-abelian category* is an exact homological category with binary coproducts.

Theorem 6.10. The category $\mathrm{Mod}(\Gamma_{\mathrm{homo}})_{\mathrm{ex/reg}}$ is exact and homological. Moreover, each small semi-abelian category $\mathbb{C}$ admits a faithful embedding

$$\mathbb{C} \hookrightarrow \mathrm{Mod}(\Gamma_{\mathrm{homo}})_{\mathrm{ex/reg}}$$

which preserves and reflects finite limits, isomorphisms and coequalisers of equivalence relations.

However, this is not yet a good embedding theorem for semi-abelian categories since we do not know whether $\mathrm{Mod}(\Gamma_{\mathrm{homo}})_{\mathrm{ex/reg}}$ has binary coproducts.

7 Future work

To conclude, we list some questions raised by this research and left for future investigations.

- This paper treats about the properties of having (M, X)-closed relations (which include many categorical properties as examples) and being protomodular; both of them giving rise to Mal'tsev conditions in the (essentially) algebraic world. Can one find a common framework to these properties? A first step in that direction can be found in [29] where pointed protomodularity has been characterised via a Horn formula of extended matrices.

- In view of the quite complex constructions of $\mathrm{Mod}(\Gamma_{(M,X)})$ and $\mathrm{Mod}(\Gamma^{\mathcal{T}}_{\mathrm{proto}})$, one could want to try to prove similar embedding theorems involving simpler categories. As shown at the end of Section 5 with the property (P_{sub}), it seems no algebraic categories will fit our needs. We can however try to get simpler essentially algebraic categories or categories of another kind. We remind the reader here that we wanted our representative categories $\mathrm{Mod}(\Gamma_{(M,X)})$ and $\mathrm{Mod}(\Gamma^{\mathcal{T}}_{\mathrm{proto}})$ to share the same properties as the categories we want to embed in. In particular, we wanted $\mathrm{Mod}(\Gamma_{(M,X)})$ to be regular with (M, X)-closed relations.

- As they are stated, Embedding Theorems 6.7 and 6.9 cannot be used in an easy way to prove results in a exact context. To achieve that, one needs to have an easy description of $\mathrm{Mod}(\Gamma_{(M,X)})_{\mathrm{ex}\,/\,\mathrm{reg}}$, or, in general, of the exact completion of a regular essentially algebraic category. If one achieve to describe these completions, we could then decide whether $\mathrm{Mod}(\Gamma_{\mathrm{homo}})_{\mathrm{ex}\,/\,\mathrm{reg}}$ has binary coproducts and then have a good embedding theorem for semi-abelian categories. One could also try to characterise in the same style as in Proposition 3.2 the essentially algebraic theories which has an exact model category. This would give another perspective to Theorem 19 in [16] which states that for a small finitely complete category $\mathbb{C}$, the essentially algebraic category $\mathrm{Lex}(\mathbb{C}, \mathrm{Set})$ is exact if and only if $\mathbb{C}^{\mathrm{op}}$ is 'pro-exact'.

- The Embedding Theorem 4.10 embeds only regular protomodular categories with some colimits in the representative category. Colimits are only required to prove that the free cofiltered limit completion is also protomodular. Is it possible to get rid of this assumption about colimits? One could try for instance to express regular protomodularity in an 'unconditional' way in the sense of [24].

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